Fast Shortest Path in Graphs with Sparse Signed Tree Models and Applications

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DIMAG, IBS

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Setting

We consider algorithmic problems that are clearly in **P**:

Binary Matrix Multiplication

Compute MN for M a binary matrix, N an arbitrary matrix.

$O(n^\omega)$ in general ($\omega < 2.371339$)

All Pairs Shortest Path (APSP)

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Assuming some nice structure and compact encoding for the graph/matrix, can we improve the complexity?

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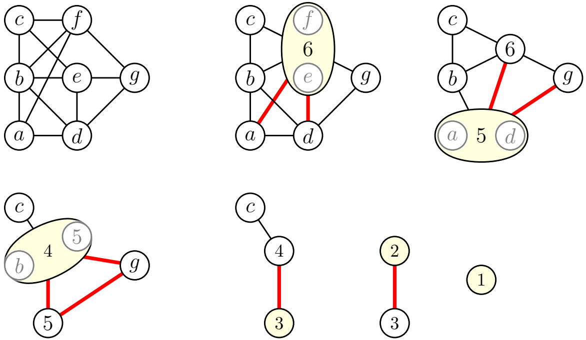
- motivated by intractable problems (FO model checking),



(Figure by Édouard Bonnet)

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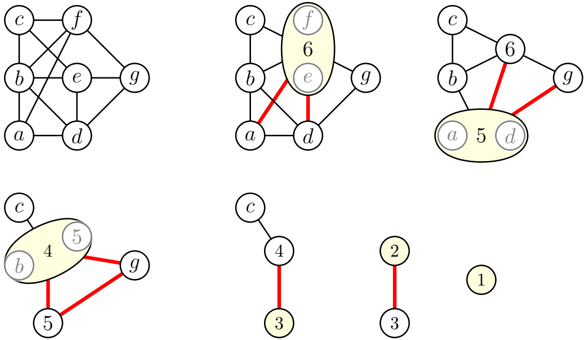
Twin-width tww(G) [BKTW '20]:

- motivated by intractable problems (FO model checking), but also
- shortest paths [BGKTW '21], [Bannach, Marwitz, Tantau '24],
- matrix multiplication [Bonnet, Giocanti, Ossona de Mendez, Thomassé '23].



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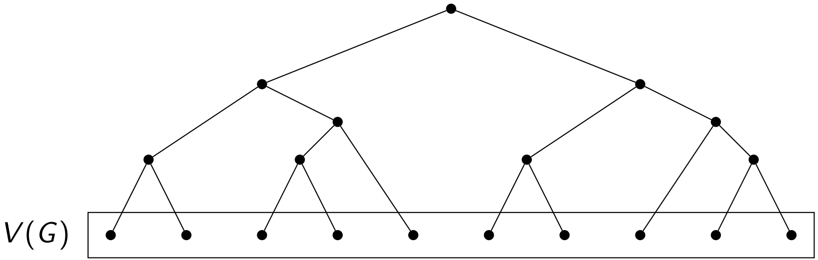

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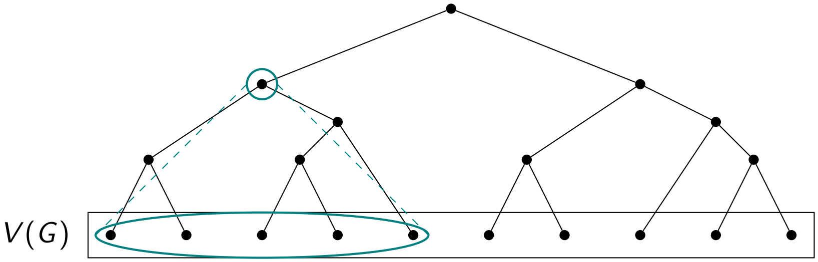

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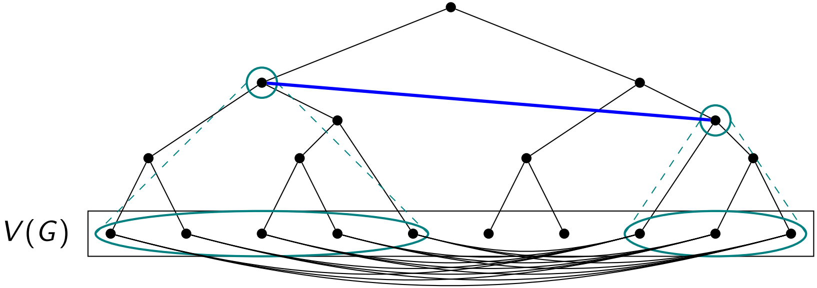

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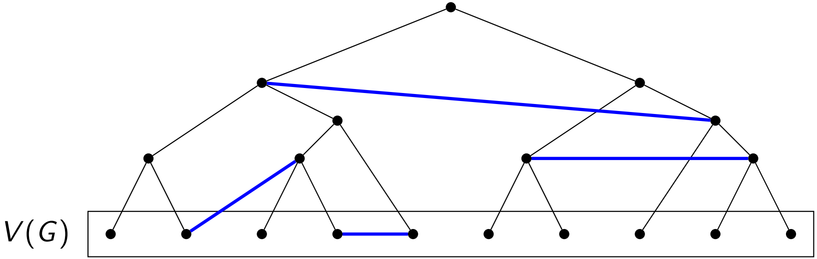

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Lemma (BNOST '21)

Given a witness that $\mathrm{tww}(G) = t$ *(a contraction sequence), one can compute a tree model with* $O(tn)$ *total size.*

But finding contraction sequences is hard!

Tree Models and Shortest Paths

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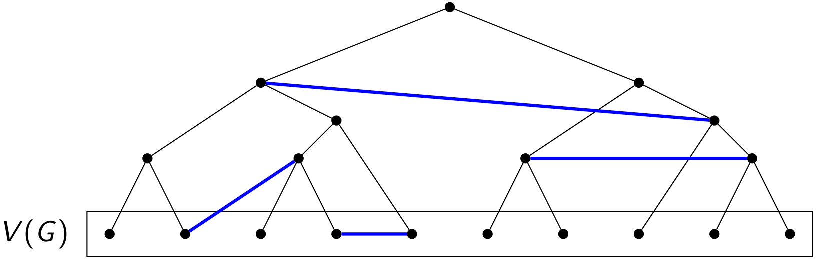

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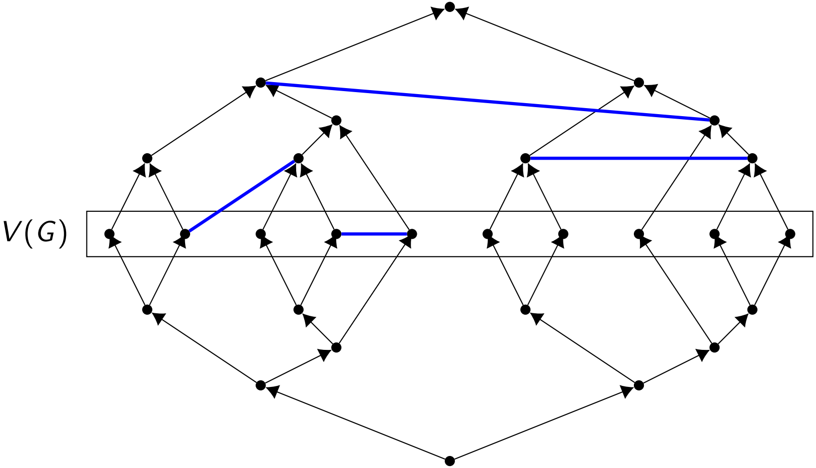

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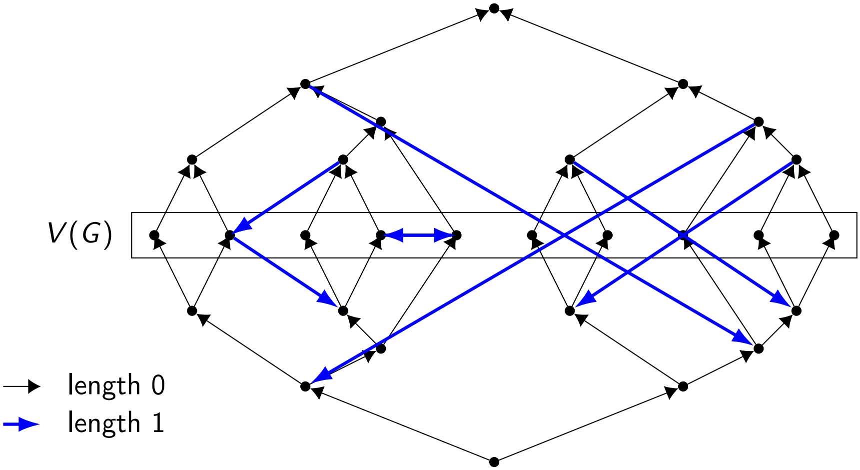
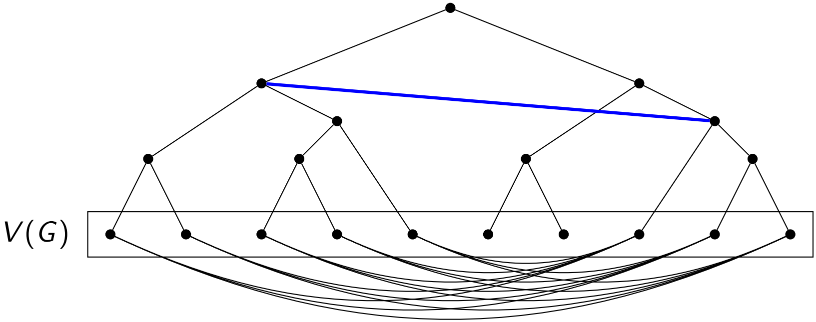


Distances here are the same as in $G \implies$ solve APSP in $O(n \cdot \text{tree model size})$.

Signed Tree Models

[Bonnet, Duron, Sylvester, Zamaraev '24]

Idea: add negative edges acting as masks



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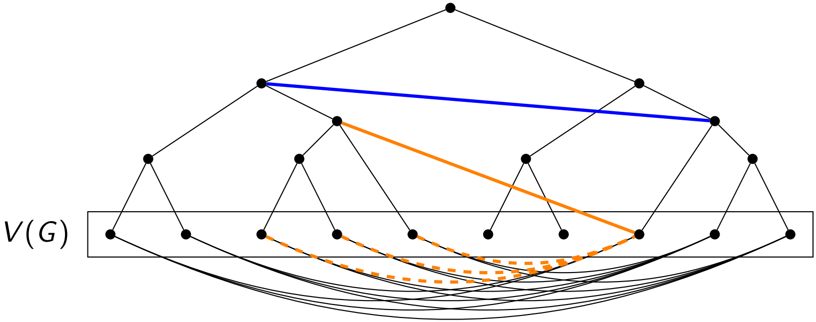

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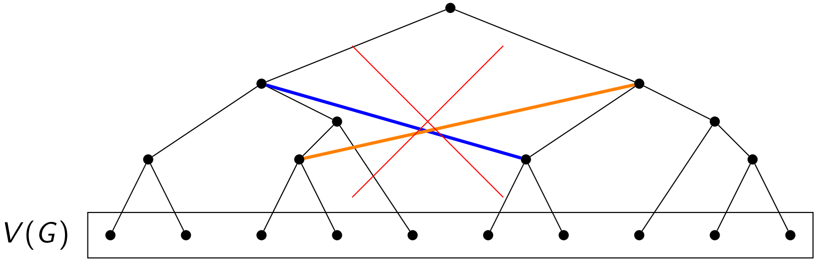

Non crossing condition: If $xy, x'y'$ are transversal edges, then

x descendant of x' and y' descendant of y is forbidden.

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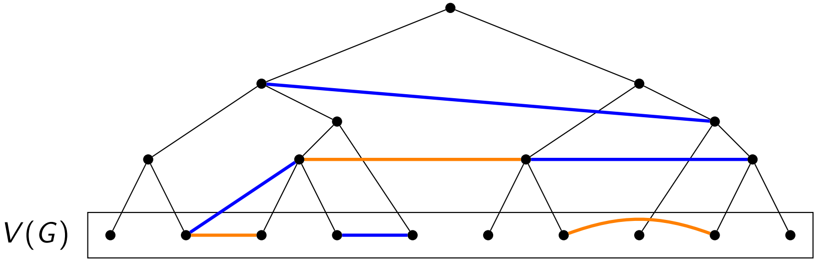

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$$x \text{ descendant of } x' \quad \text{and} \quad y' \text{ descendant of } y \quad \text{is forbidden.}$$

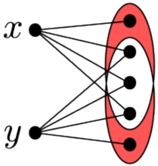
$$xy \in E(G) \quad \Longleftrightarrow \quad \text{lowest transversal edge covering } xy \text{ is positive}$$

Finding Signed Tree Models

[Alecu, Atminas, Lozin '21] G has *symmetric difference* $\leq t$ if

$$\exists x, y \in V \text{ such that } |N(x) \triangle N(y) \setminus \{x, y\}| \leq t$$

and the same on all induced subgraphs



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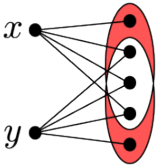

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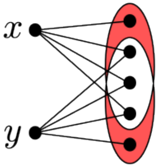

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Lemma (Bonnet, Duron, Sylvester, Zamaraev '24)

For graphs with symmetric difference t, one can compute a signed tree model of size $O(tn)$ in time $O(n^3)$.

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Signed tree models are easier to compute, capture more graphs

Positive tree models (i.e. not signed) can be used in shortest path problems

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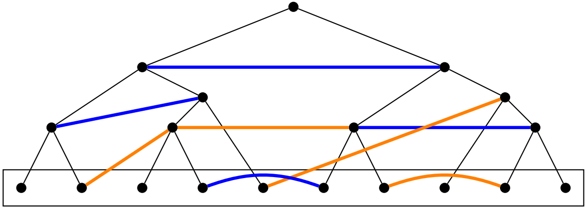
Theorem (Bonnet, G., Kim, Moon)

One can convert a signed tree model with p transversal edges into a positive tree model with $p \log^2 n$ transversal edges in time $O(p \log^2 n)$.

Proof Sketch (1)



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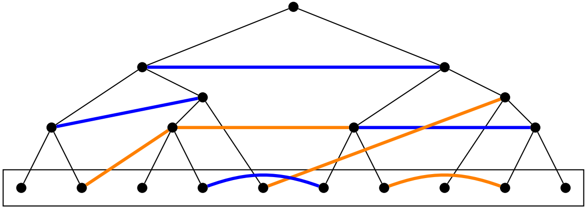
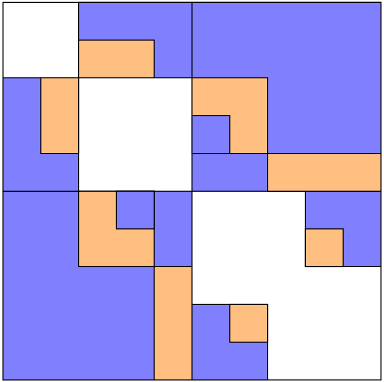
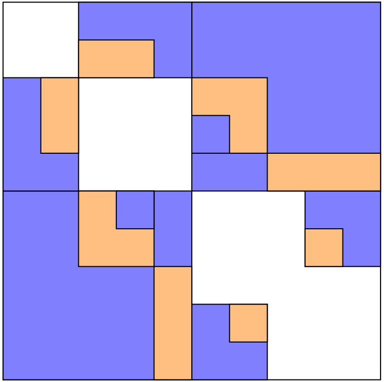




In the adjacency matrix (with left-to-right ordering from the tree):

- transversal edge in signed tree model $\rightsquigarrow$ rectangle in the matrix
- non-crossing condition $\rightsquigarrow$ rectangles are a laminar family

Proof Sketch (2)



Put the rectangles on the matrix (smaller on top)

blue = edge   orange or empty = non-edge

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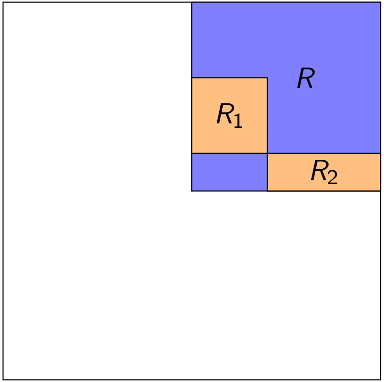

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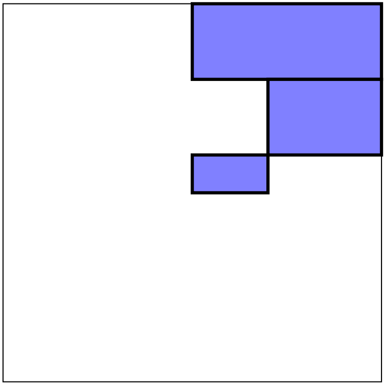

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If $R_1, \ldots, R_k$ are disjoint rectangles in R, one can partition $R \setminus (R_1 \cup \cdots \cup R_k)$ into $O(k)$ rectangles.

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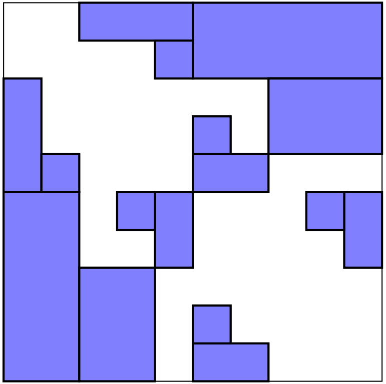

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Repeat for all blue rectangles to partition the adjacency matrix.
(aka Interval Biclique Partition)

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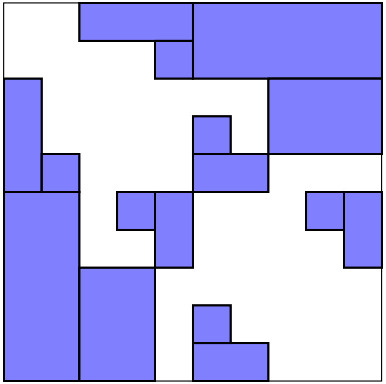

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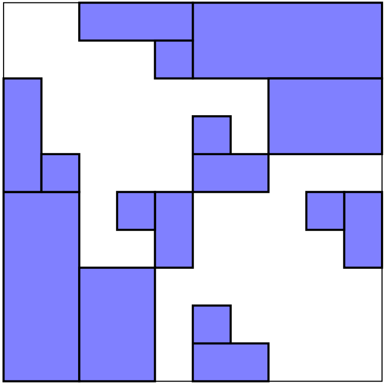

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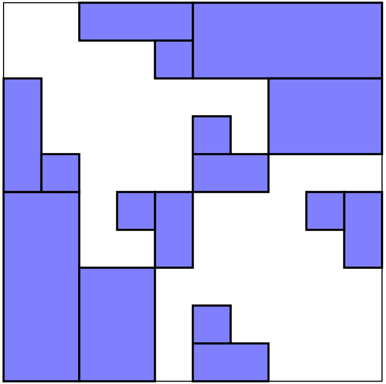

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- Any rectangle can be represented by $O(\log^2 n)$ edges on a balanced binary tree
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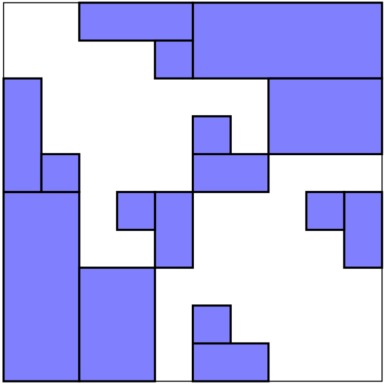

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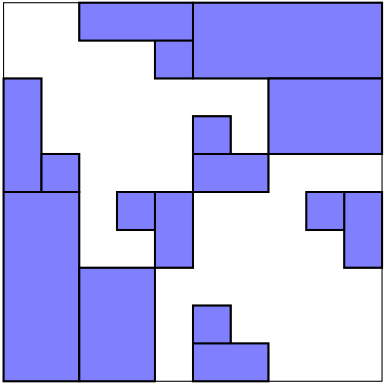

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- Can also be used for matrix multiplication in time $O(pn)$ [this work] [Kozma, Opler '26]

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For graphs of bounded symmetric difference:

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Also: $\tilde{O}(n^{7/3})$ for bounded symmetric difference
(same algorithm with different parameter tuning).

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Previous and Newer Works

For matrix multiplication MN:

- $O(n^2 \log n)$ for bounded twin-width [Bonnet, Giocanti, Ossona de Mendez, Thomassé '23] when given a witness of twin-width for both M and N
- $O(n^2 \log^2 n)$ for M with linear neighbourhood complexity but N arbitrary
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