Transducing Linear Decompositions of Tournaments

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DIMAG, IBS

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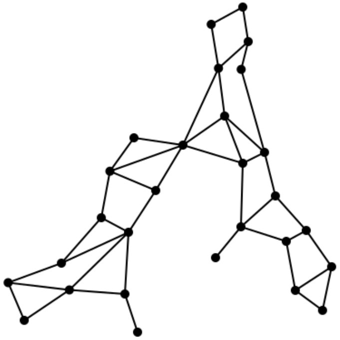
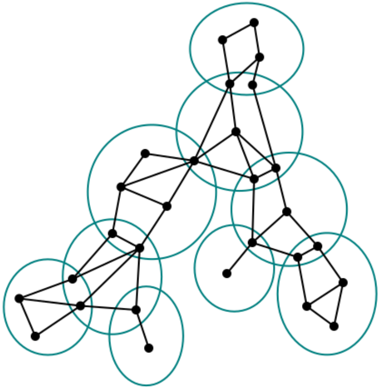
Tree-width



Small tree-width graph G

"Looks like a tree"

Tree-width



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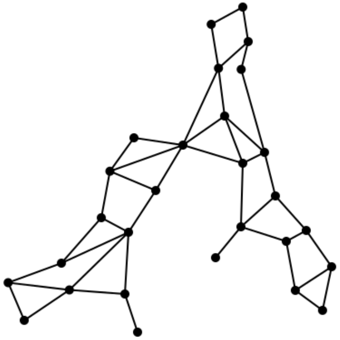

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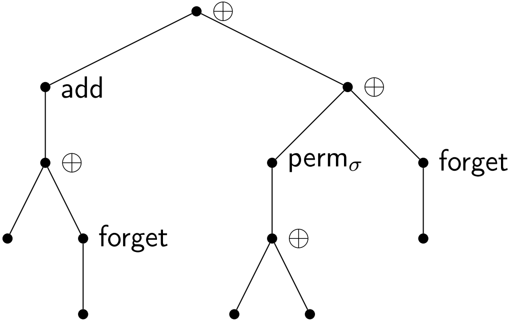

Tree decomposition

Labelled tree encoding G

Graphs and tree-decompositions

How to go back and forth?

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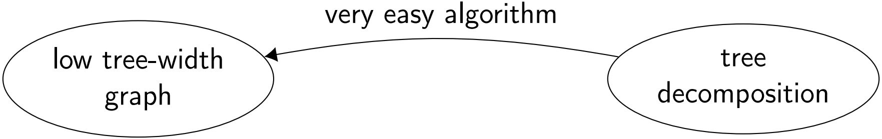

Graphs and tree-decompositions

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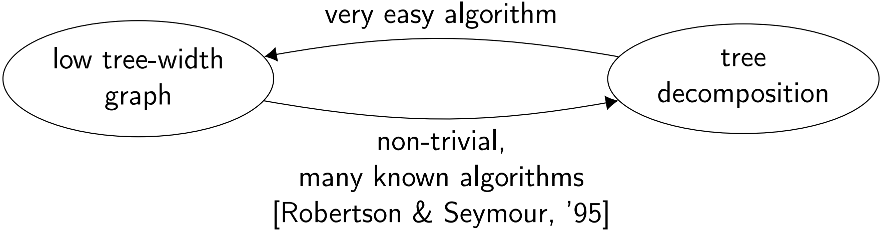

MSO Interpretations

MSO logic: (in the case of graphs)

- $\forall x, \exists y, \forall X, \exists Y$: quantifiers on elements (vertices) and *sets*
- $\neg, \vee, \wedge$: boolean operators
- $E(x, y)$: edge predicate
- $x = y, x \in X$

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MSO interpretation $\Phi = (\phi_V; \phi_1, \ldots, \phi_k)$

Given input G, define $\Phi(G)$:

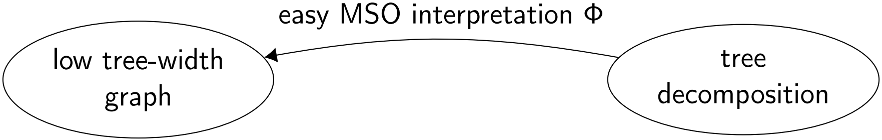
- universe (vertices)

$$\{x \in V(G) : G \models \phi_V(x)\}$$

- for $i = 1, \ldots, k$, add the relation

$$\{(v_1, \ldots, v_r) : G \models \phi_i(v_1, \ldots, v_r)\}$$

Courcelle's Theorem



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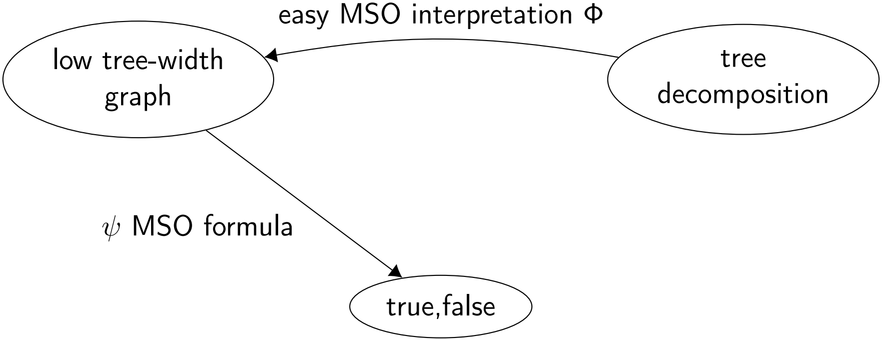

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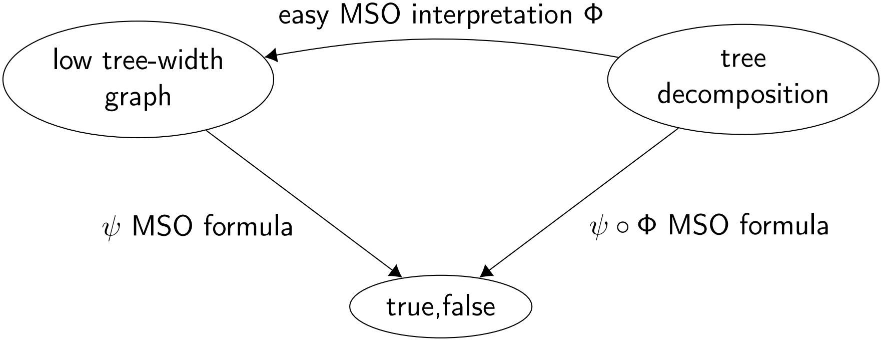

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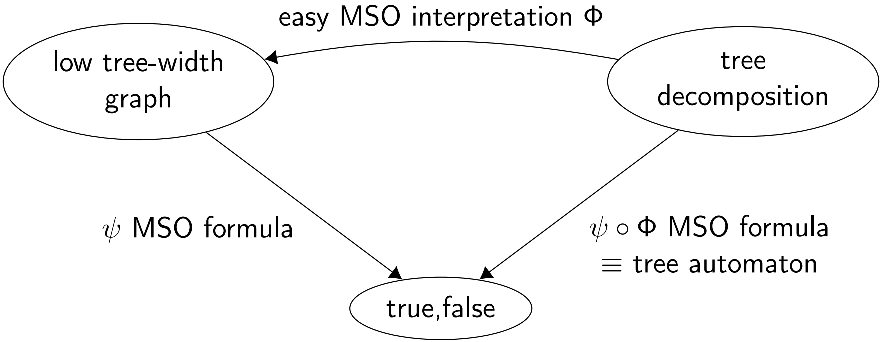

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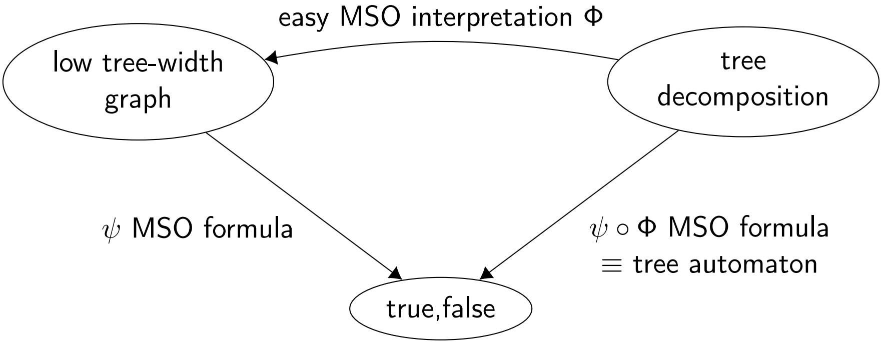
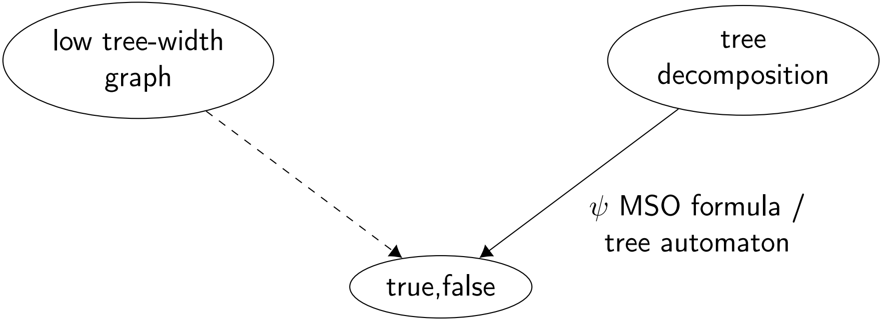


Theorem (Courcelle's theorem)

Any MSO formula on graphs is equivalent to a tree automaton on tree decompositions (for fixed tree-width).

In the other direction



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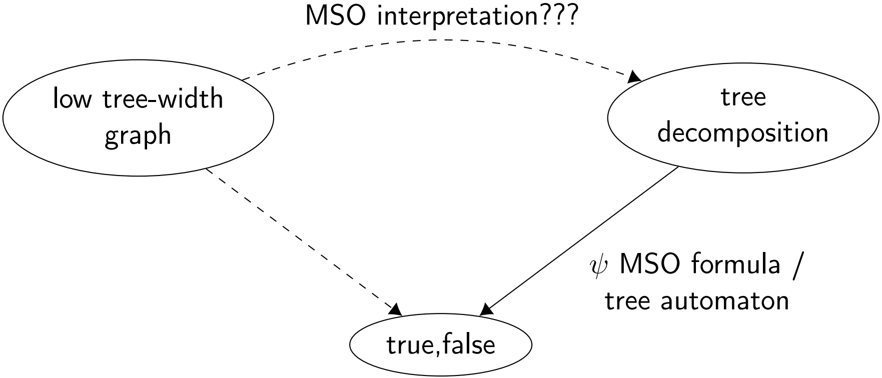
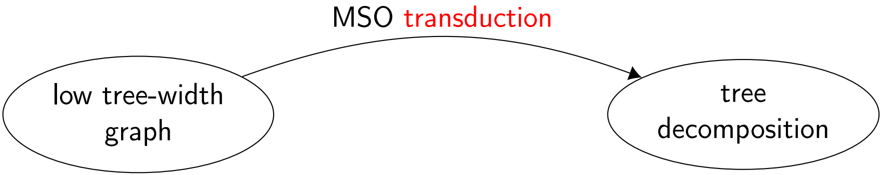


Definable Tree Decompositions



Theorem (Bojańczyk & Pilipczuk '16)

For any k, there is an MSO interpretation Φ such that for any graph G of tree-width k, there is a way to add vertex colours to G so that $\Phi(G)$ is a tree decomposition of G of width $f(k)$.

MSO transduction

MSO transduction Φ: given G

- Consider all possible vertex colourings $\lambda : V(G) \to [k]$.
  This is non-deterministic! $\Phi(G)$ *is a set of structures.*
- Apply an MSO interpretation (where formulas can talk about the colours λ)

And some other operations which are not that important here.

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The proof has two parts:

1. Transduce decompositions on graphs of small path-width.
   Uses Simon's Factorisation to reduce to 'very regular' cases.

2. 'Clever' reduction from tree-width to path-width using reasoning on paths.

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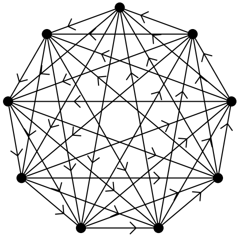
   Generalisations:
   - linear clique-width for dense graphs [Bojańczyk, Grohe, Pilipczuk '18],
   - linear rank-width for some matroids [Campbell, Guillon, Kanté, E.J. Kim, Oum '25]
2. 'Clever' reduction from tree-width to path-width using reasoning on paths.
   No known generalisation.

Tournaments

Tournament: directed graph where for all vertices $u \neq v$, exactly one of $u \to v$ or $u \leftarrow v$ is an edge.



Linear Clique-width

Object: coloured tournament $T = (V, E, \lambda)$, $\lambda : V \to [k]$.
Operations:

Add For some specified $c \in [k]$, $N \subseteq [k]$,

- create a new vertex v
- define $\lambda(v) := c$
- for any existing vertex u, add $v \to u$ if $\lambda(u) \in N$, and $v \leftarrow u$ otherwise.

Recolour Set $\lambda := \lambda \circ f$ for some $f : [k] \to [k]$.

$\simeq k^k$ operations in total.

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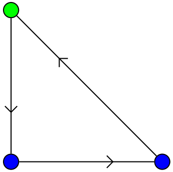
T has *linear clique-width* $\leq k$ if it can be created by these operations.
The sequence of operations constructing T is a *linear decomposition*.

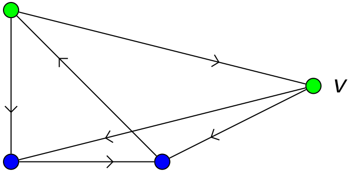
For k fixed, a linear decomposition is a word over some fixed alphabet.

Linear Clique-width – Example



Linear Clique-width – Example

Add a vertex v with colour green
and edges $v \to$ blue and $v \leftarrow$ green.



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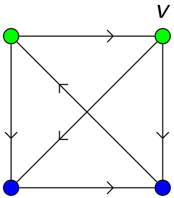
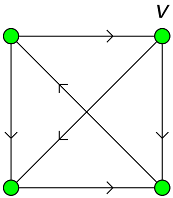


Linear Clique-width – Example

Recolour blue to green



Linear Clique-width – Example

Add a vertex w with colour blue
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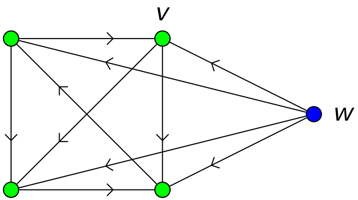

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FO definable decompositions for tournaments

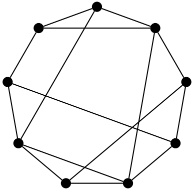
Theorem (G., Ghasemi, Kanté)

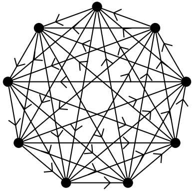
For any k, there is an FO transduction Φ such that for any tournament T of linear clique-width k, given G, Φ produces a linear decomposition of G of width $f(k)$.

- First-order (FO) logic is enough (no set quantifiers)
- The output is always a *linear* decomposition (= encoding of T as a word).
  This is specific to tournaments, and impossible for bounded path-width graphs!
- But it only applies to tournaments with small linear clique-width (= encoded by words), not clique-width in general (= encoded by trees).

Why Tournaments

Graph



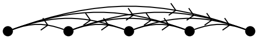


Tournament

Ramsey Theorem

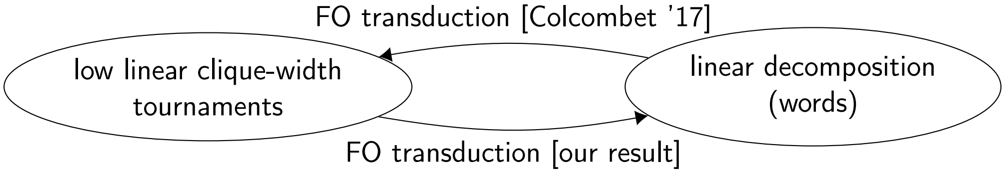
Ramsey Theorem

clique/stable set
= no structure



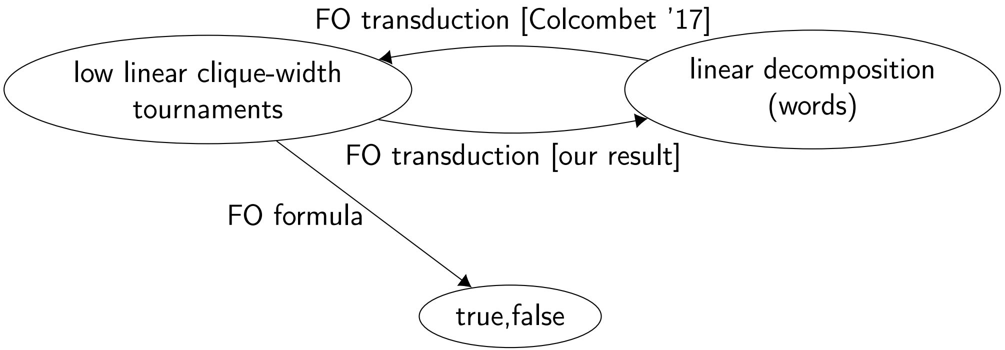
transitive tournament
= total ordering

Existential MSO



Linear decomposition / words encoded as ordered structures (this matters for FO logic)

Existential MSO



FO transductions and FO formulas cannot be composed.

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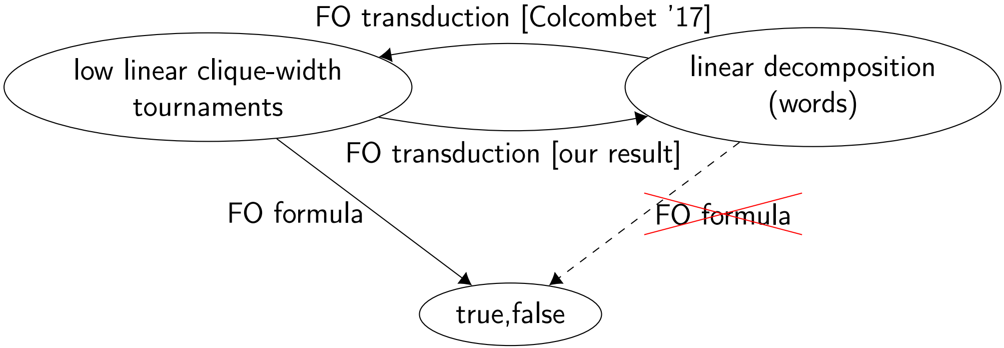

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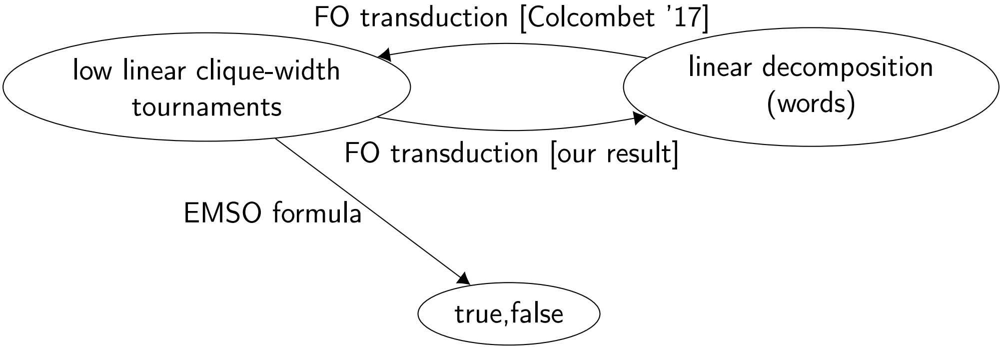

FO transductions and EMSO formulas can be composed!
EMSO: $\exists X_1, \ldots, \exists X_k\ \psi$ (with ψ FO formula)

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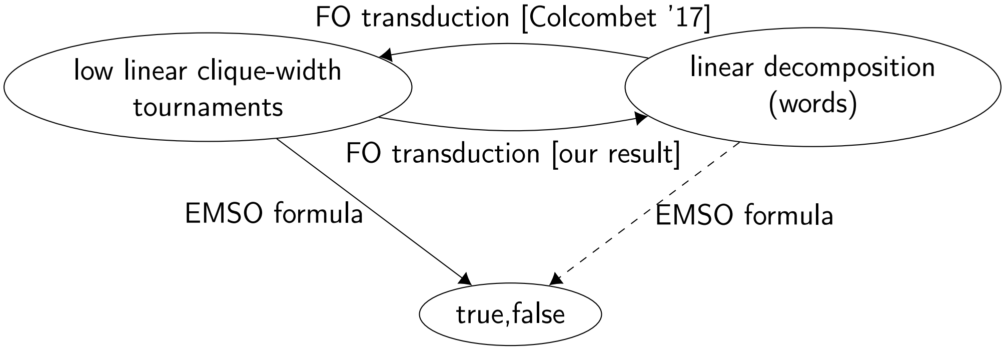

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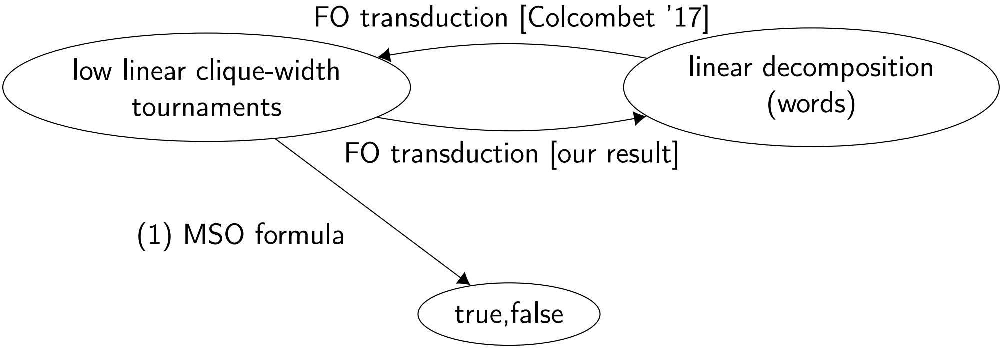

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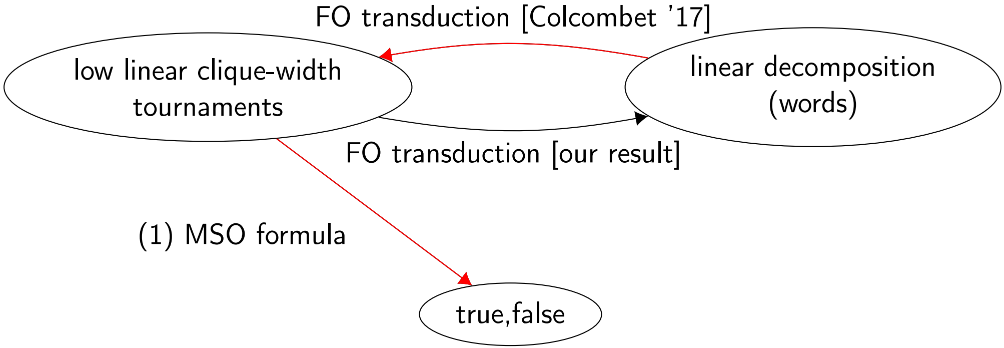

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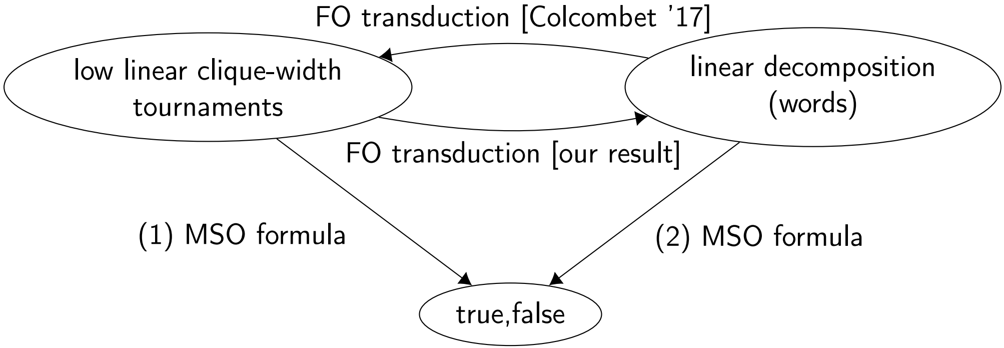

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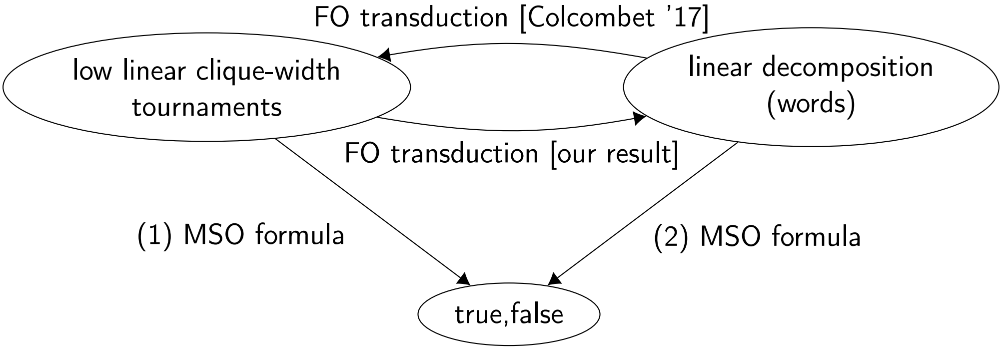
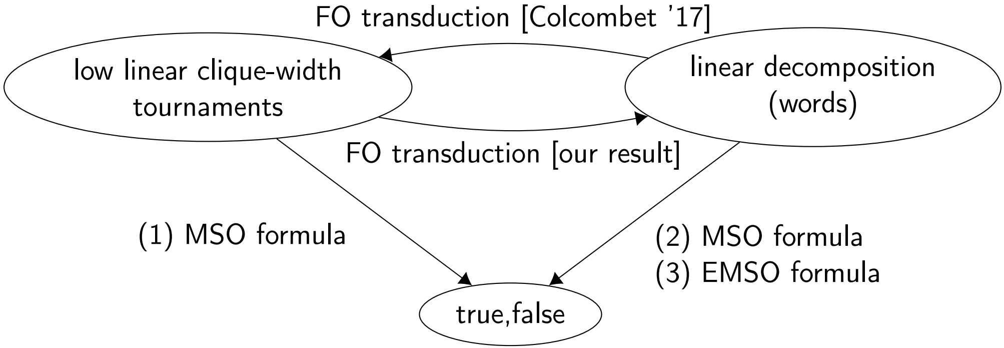


Büchi–Elgot–Trakhtenbrot: MSO and EMSO are equivalent on words

Existential MSO



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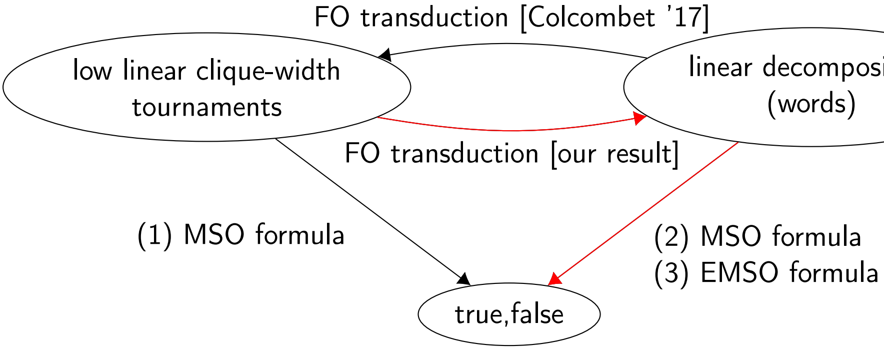

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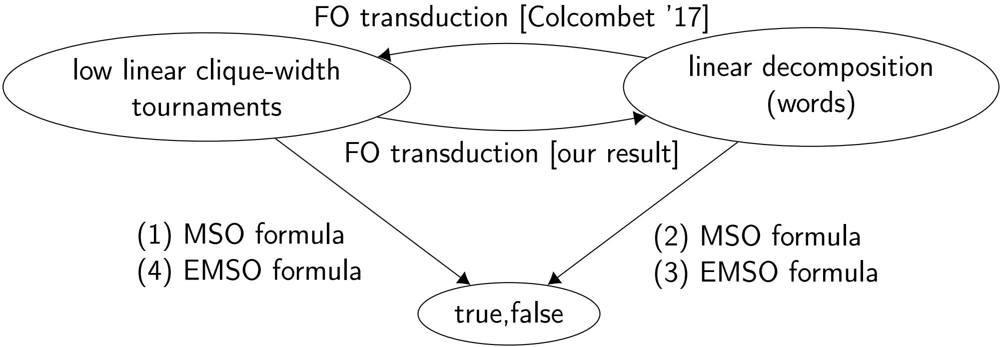

Corollary (G., Ghasemi, Kanté)

For bounded linear clique-width tournaments, MSO and EMSO have the same expressive power.

Proof idea

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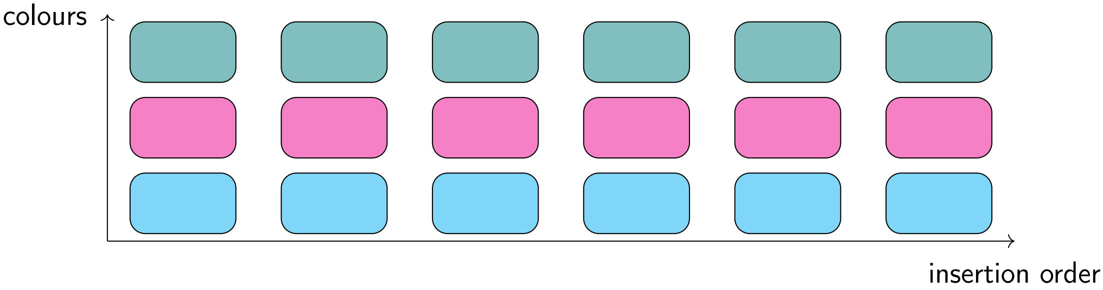
- There is some (hidden) linear decomposition D of G, which we want to describe.
- Actual goal: describing the order in which vertices of T are added.
  (Adding the rest of D afterwards is easy.)
- Big hammer: Simon's Factorisation allows to pretend that D is 'very regular'.

Simon's Factorisation Magic (grossly oversimplified)

The linear decomposition is split into intervals:

- each interval is 'strictly simpler' (bounded induction)
- vertex colours are consistent at the interval boundaries
- edges between intervals only depend on (1) the relative ordering and (2) the colour.



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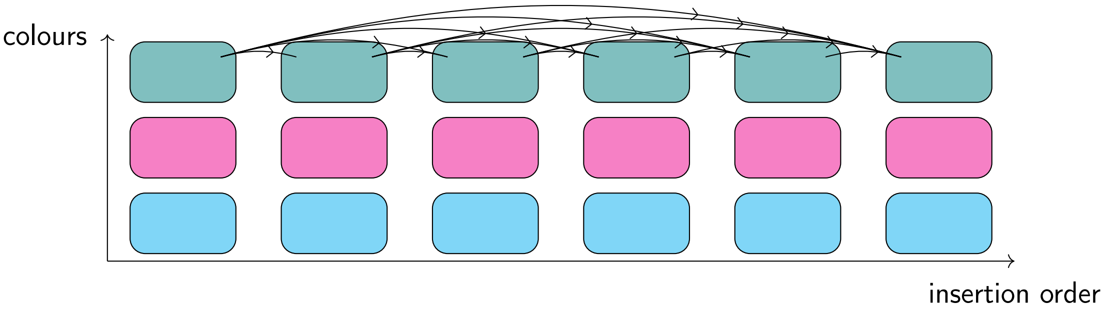

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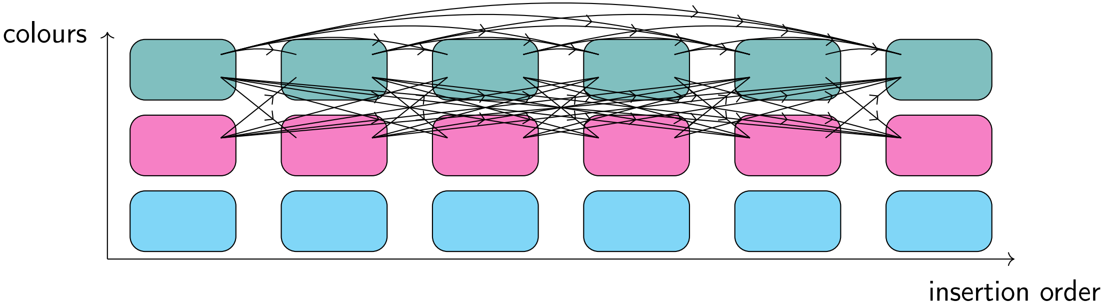

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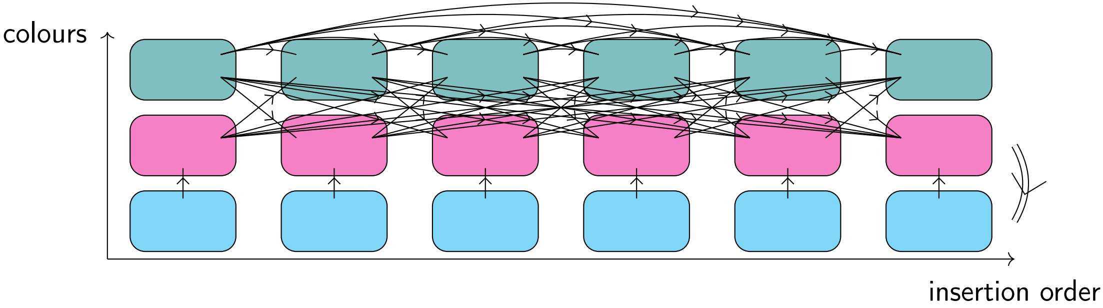
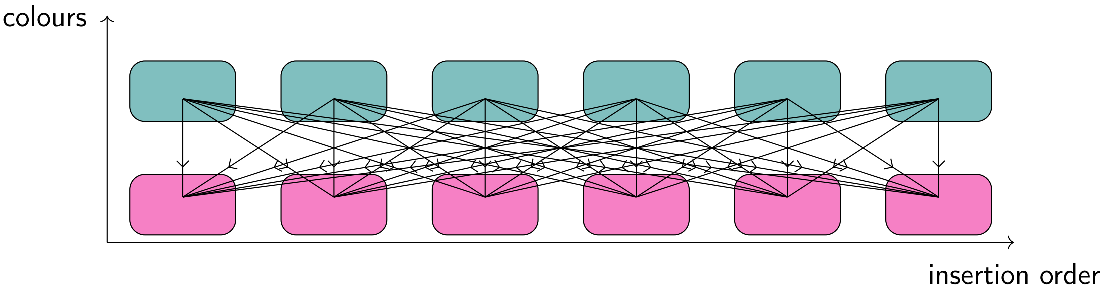


The bad case

All edges green $\rightarrow$ red:



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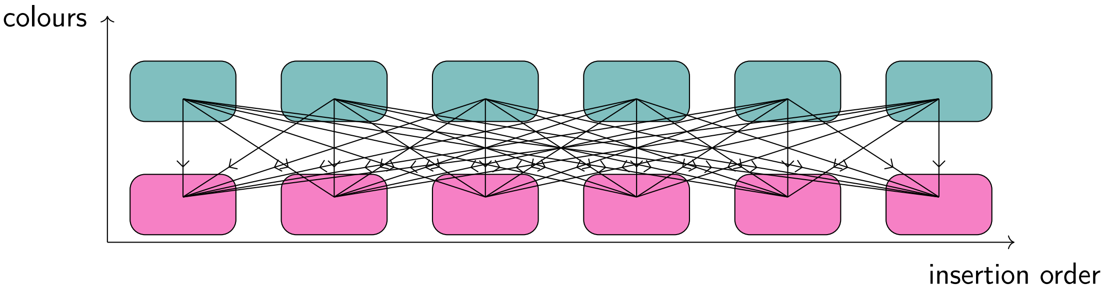
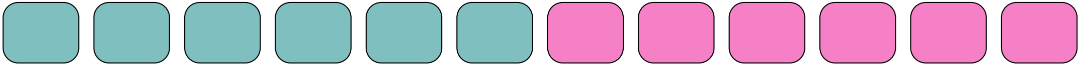


Rearrange as:



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