Cycle Space of Graphs Excluding a Minor

Colin Geniet and Ugo Giocanti

in parallel with:

B. Miraftab, P. Morin, Y. Yuditsky

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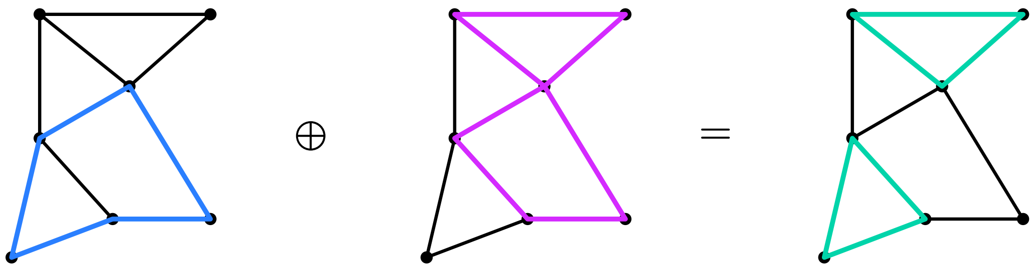
Cycle Space

$2^{E(G)}$: $\mathbb{F}_2$-vector space of subgraphs (sum $\oplus$ is symmetric difference)

cycle space $\mathcal{C}(G) \subset 2^{E(G)}$: subspace of Eulerian subgraphs (= generated by cycles)

cycle basis : basis of the cycle space



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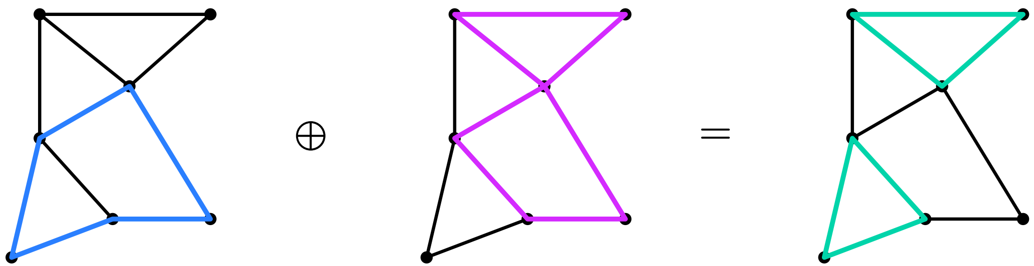

Language abuse warning: cycle $=$ eulerian subgraph

basis $=$ generating set

Mac Lane's Planarity Criterion

Theorem (Mac Lane, '37)

G is planar if and only if $\mathcal{C}(G)$ has a basis using each edge at most 2 times.

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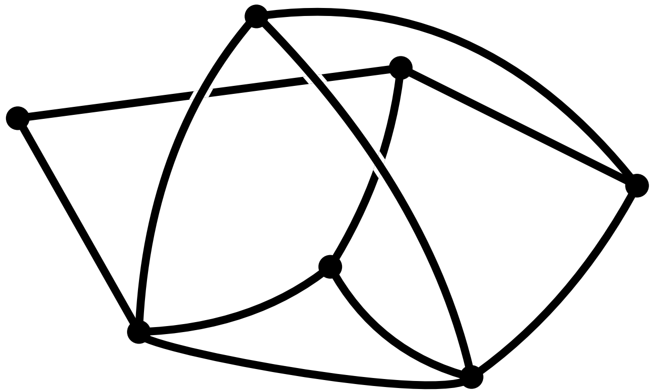
for each $C \in \mathcal{B}$, add a disk with boundary C.



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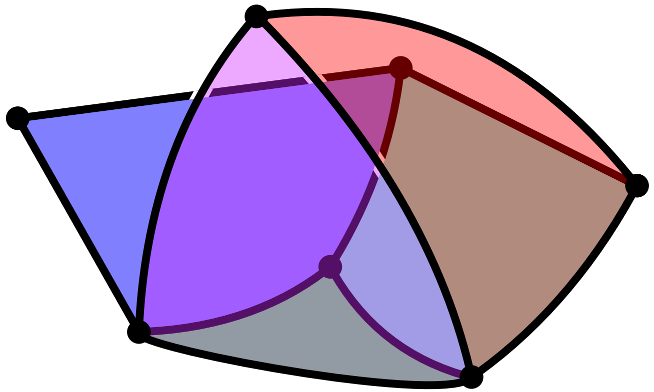

- each edge is used at most twice $\Rightarrow$ Σ is a surface
  (except vertices of G are an issue)
- $\mathcal{B}$ is a basis $\Rightarrow$ Σ is contractible

G is drawn on a contractible surface Σ, so it is planar.

Basis Number

Basis number [Schmeichel '81]

The *congestion* of a cycle basis $\mathcal{B}$ is the maximum number of uses of an edge in $\mathcal{B}$:

$$\max_{e \in E(G)} \#\{C \in \mathcal{B} : e \in C\}$$

The *basis number* of G ($\mathrm{bn}(G)$) is the minimum congestion of a cycle basis of G.

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- $\mathrm{bn}(K_n) = 3$
- degree ≥ 3 and girth $\geq k$ imply basis number $\geq k/2$

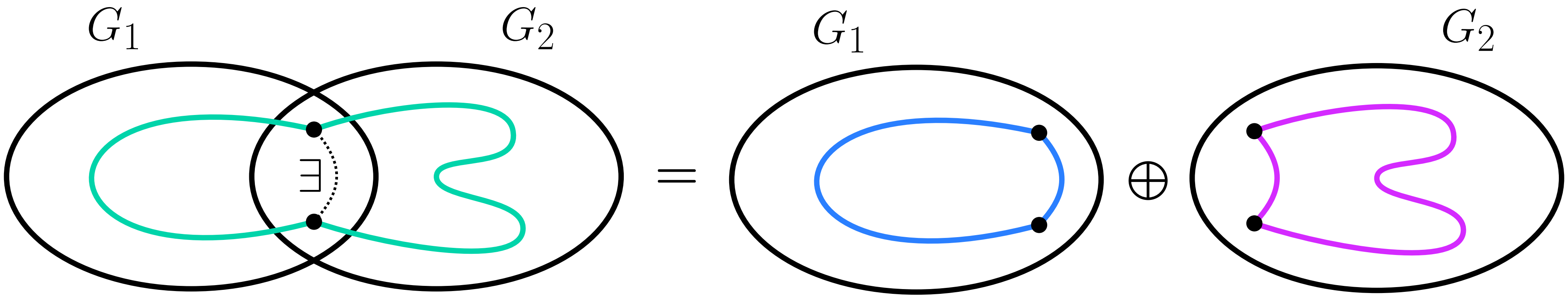
Basic Operations I: Connected Separators

$\cup, \cap$ are edge-wise

Lemma

Suppose $G = G_1 \cup G_2$ with $G_1 \cap G_2$ is connected. If $\mathcal{B}_1, \mathcal{B}_2$ are cycle bases of G_1, G_2, then $\mathcal{B}_1 \cup \mathcal{B}_2$ generates $\mathcal{C}(G)$. In particular,

$$\mathrm{bn}(G) \le \mathrm{bn}(G_1) + \mathrm{bn}(G_2).$$

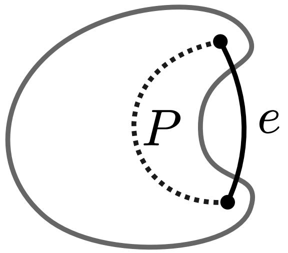


Basic Operations II: Editions

Suppose $\mathrm{bn}(G) = k$.

Edge addition

$\mathrm{bn}(G + e) \leq k + 1$



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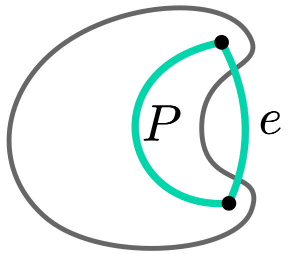

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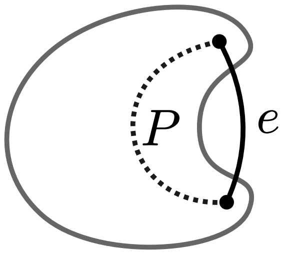

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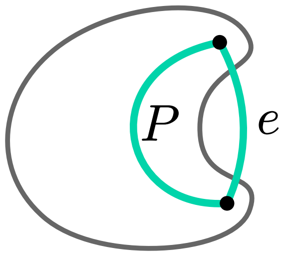
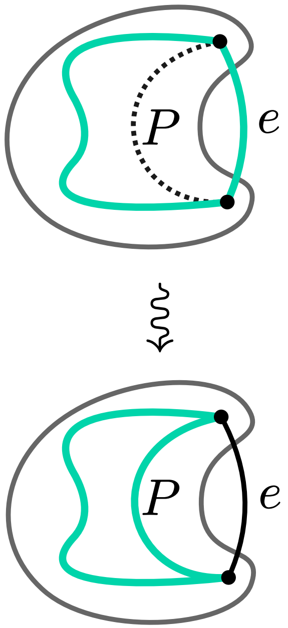


Edge deletion

$$\mathrm{bn}(G - e) \leq 2k$$



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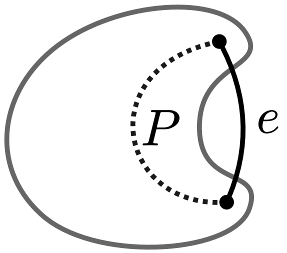

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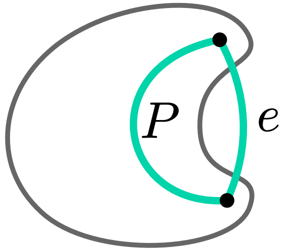

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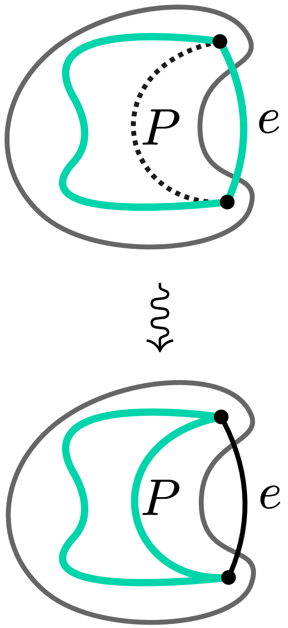
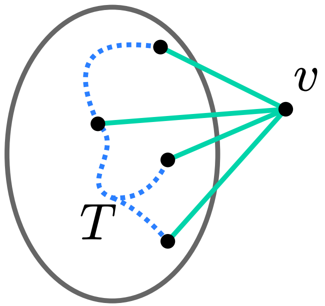


Vertex addition

$$\mathrm{bn}(G + v) \leq k + 2$$



- $G + v = G \cup (T + v)$
- $G \cap (T + v)$ is connected
- $T + v$ is planar

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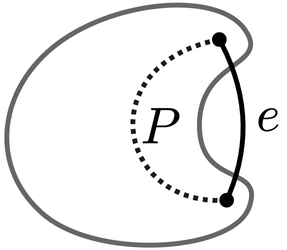

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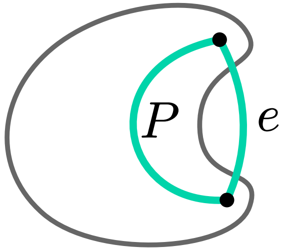

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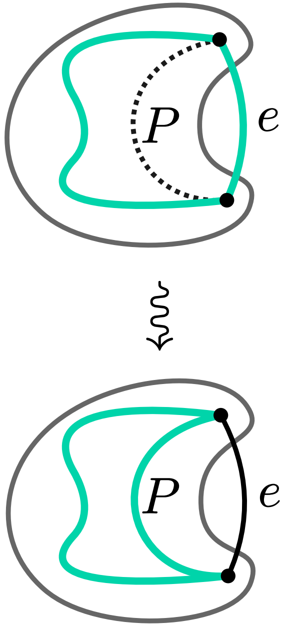

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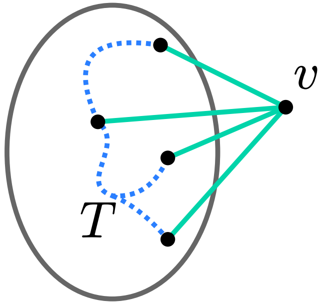

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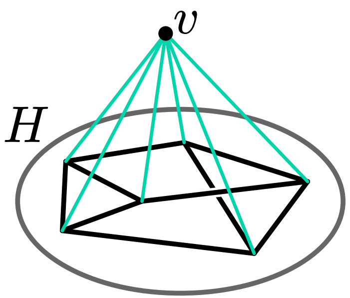
Vertex deletion

$\mathrm{bn}(G - v)$ can be unbounded!



For H cubic:

- $\mathrm{bn}(H + v) \leq 3$
- $\mathrm{bn}(H)$ unbounded

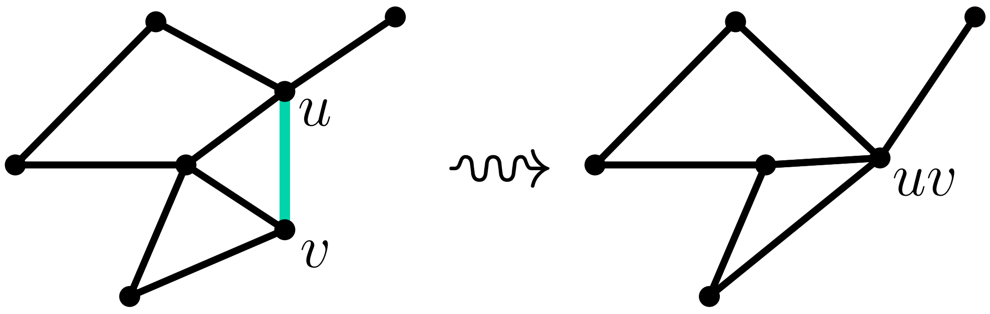
Minors

H is a *minor* of G if H is obtained from G by

- vertex deletions
- edge deletions
- edge contractions

G is H-*minor free* if G does not contain H as minor.

edge contraction:



Surfaces and Graph Minors

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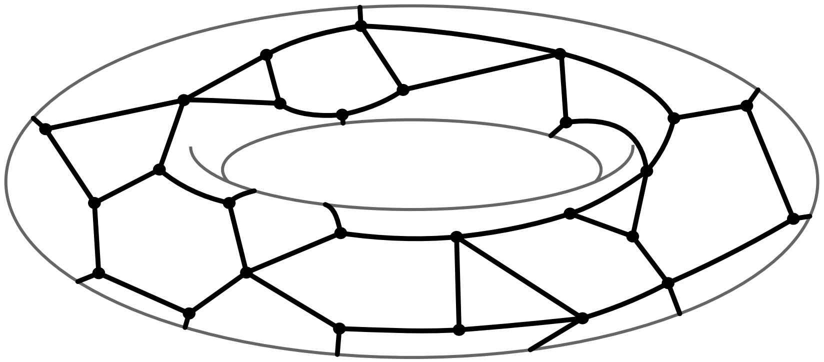
Theorem (G. & Giocanti)

If G is H-minor free, then $\mathrm{bn}(G) \leq f(H)$ for some function f.

Graph Minor Structure

[Robertson & Seymour] For fixed H, all H-minor free graphs can be constructed by:

- taking graphs on a fixed surface ✓



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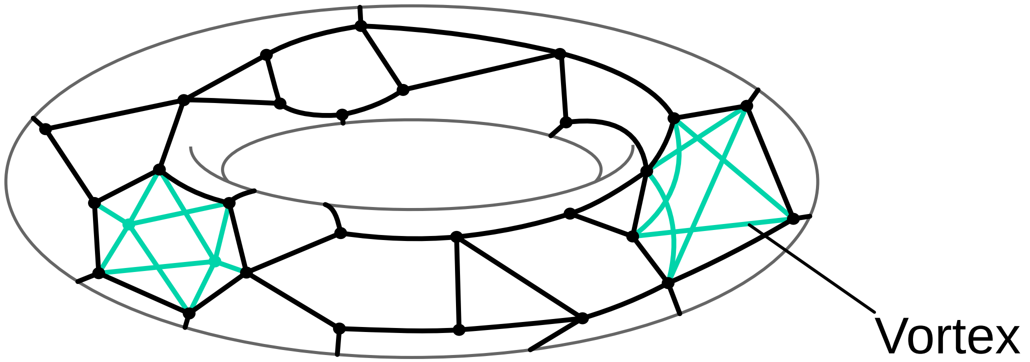

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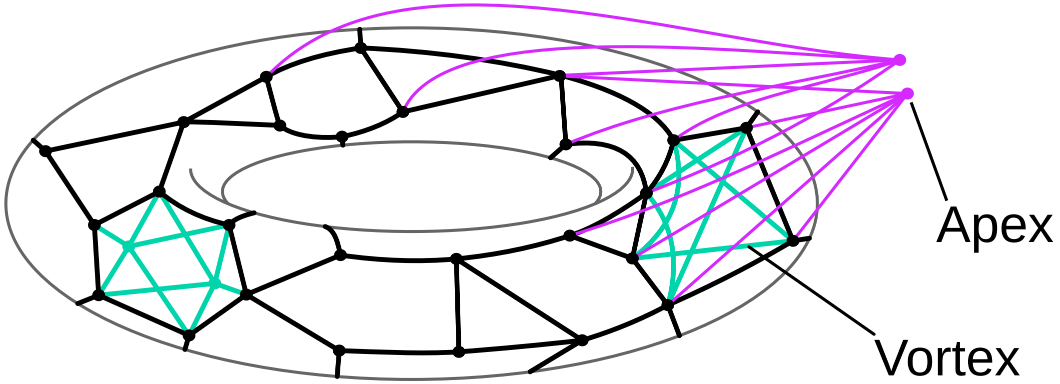

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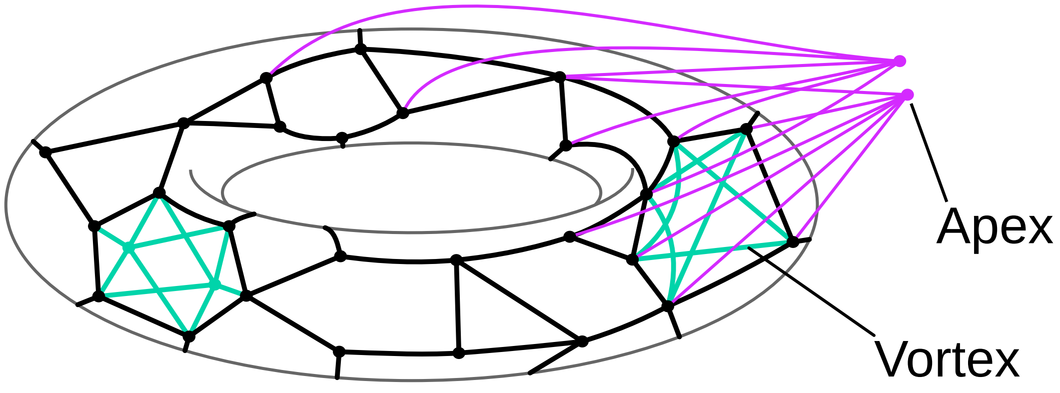
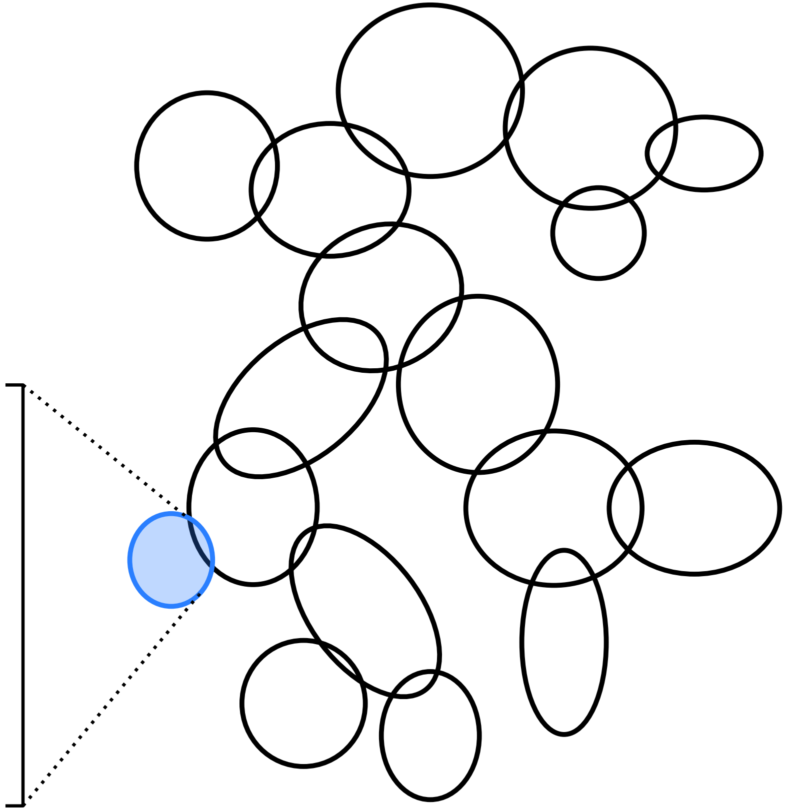
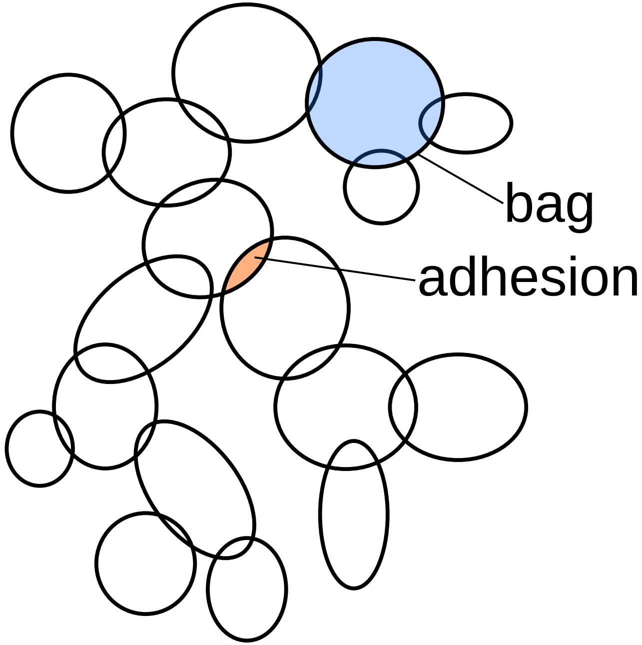
- taking graphs on a fixed surface ✓
- adding a few *vortices* of small depth ?
- adding a few *apices* ✓
- combining such graphs into a *tree decomposition* !





Tree-width



H-minor free graphs:

- each bag is a graph of small genus
  + apices and vortices
- adhesions have bounded size

bounded tree-width graphs:

- bags and adhesions have bounded size

Tree-width

bn: basis number, tw: tree-width

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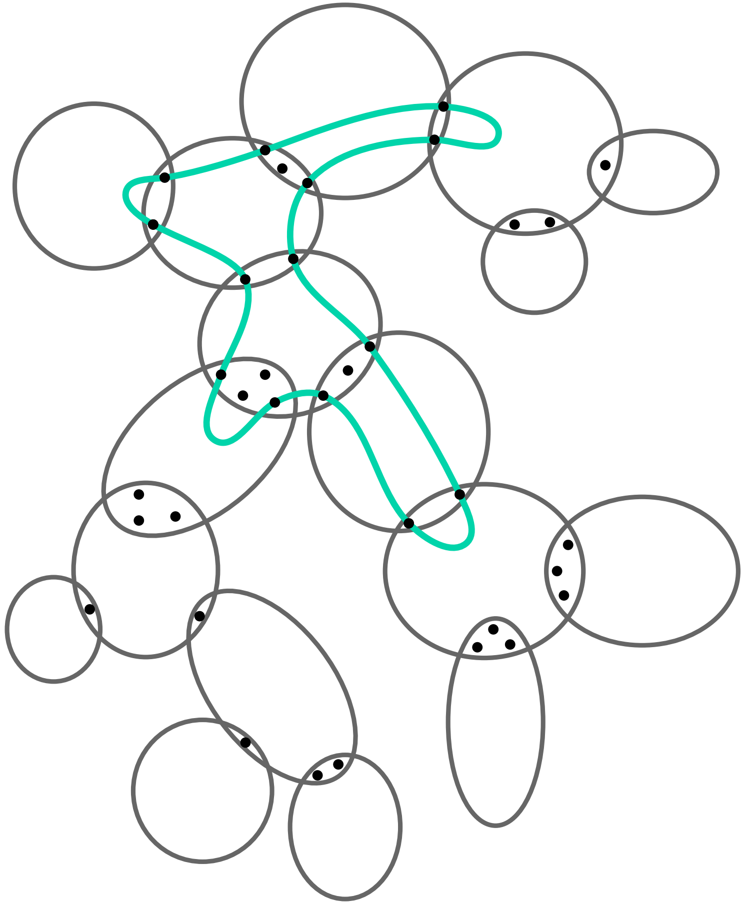
Theorem (G. & Giocanti)

There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for any G, $\mathrm{bn}(G) \leq f(\mathrm{tw}(G))$.

Proof Attempt



Take a cycle C.

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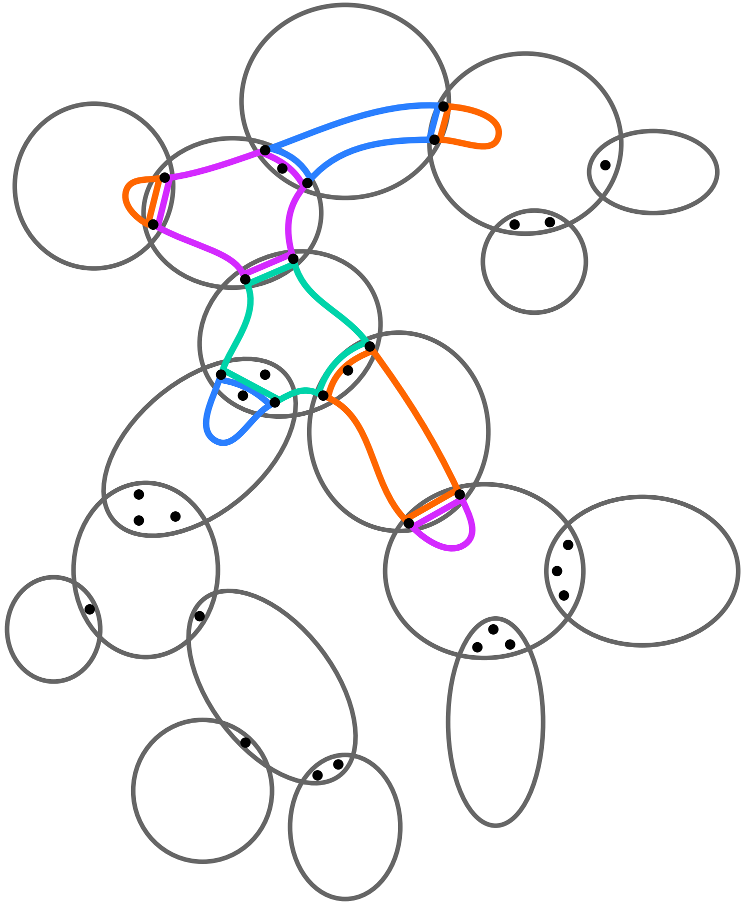

Take a cycle C.

Idea: split C into cycles $C_1, ..., C_k$, each contained in a bag.
Problem: adhesions need to be connected.

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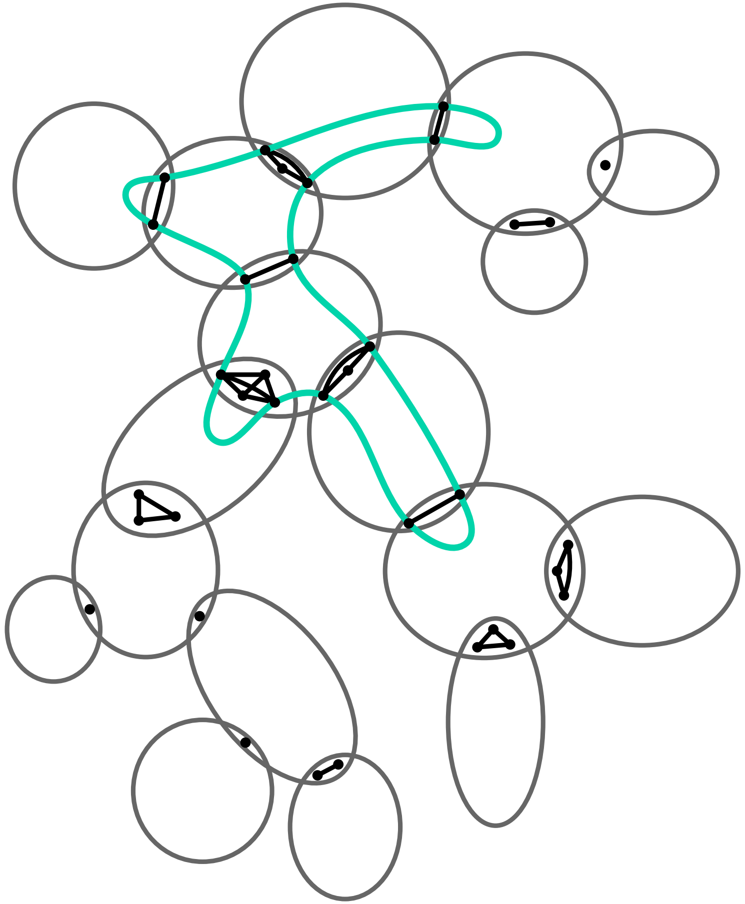

G^+: turn each adhesion into a clique.

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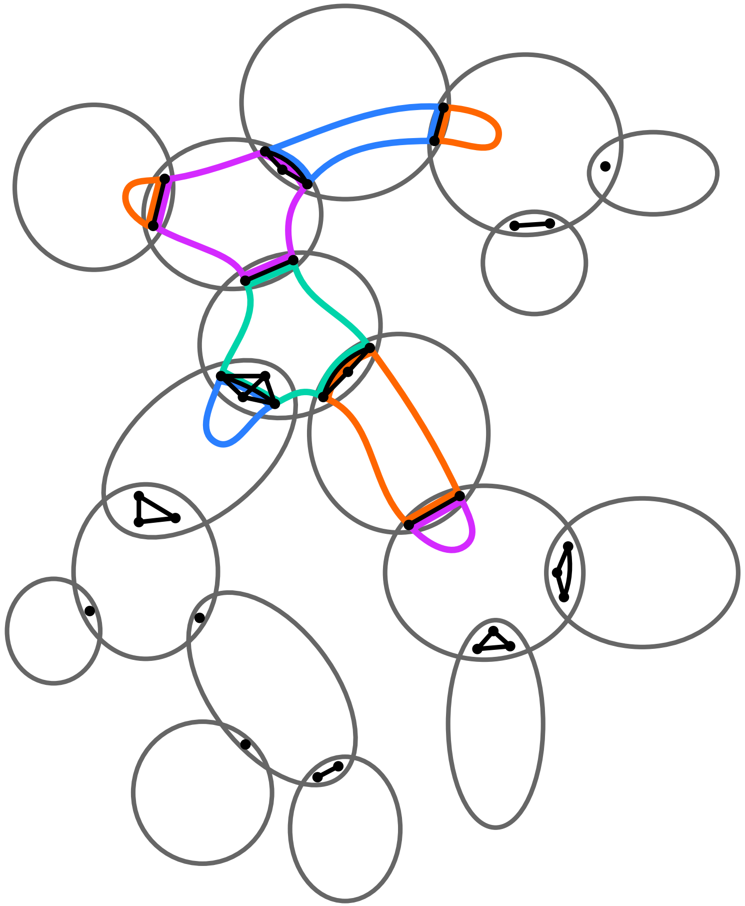

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For each bag b, pick a cycle basis $\mathcal{B}_b$ of G^+ restricted to b.

$\mathcal{B} := \bigcup_b \mathcal{B}_b$ generates $\mathcal{C}(G^+)$.

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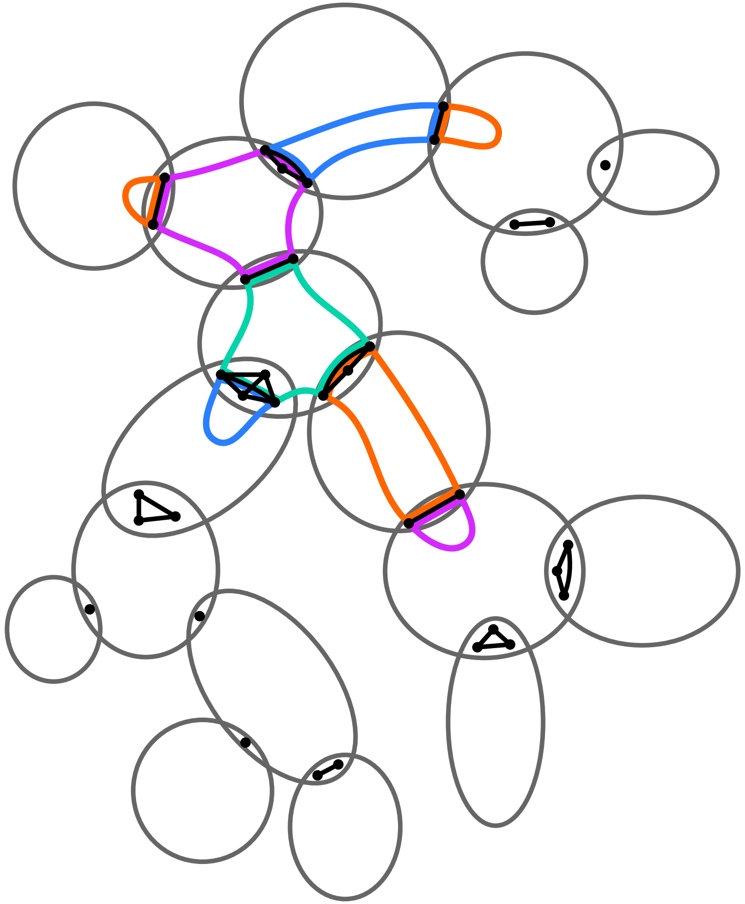

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We need to

1. get rid of 'fake edges' in $E(G^+) \setminus E(G)$
2. control the congestion

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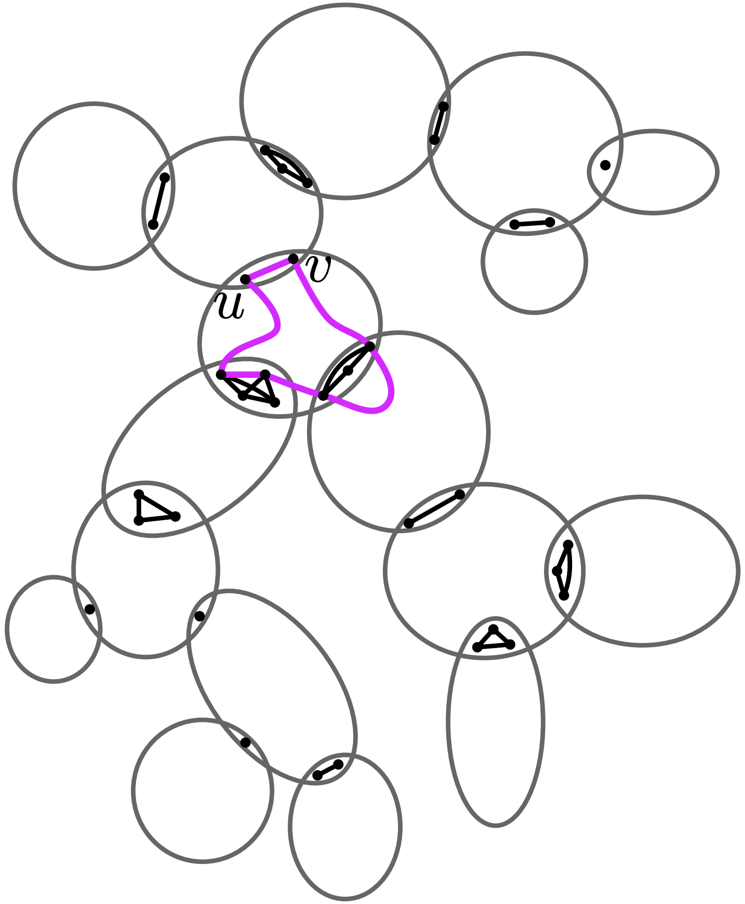

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For each 'fake edge' $uv \in E(G^+) \setminus E(G)$, pick a uv-path P_{uv} in G.

$\mathcal{B}'$ obtained by replacing each uv by P_{uv} in $\mathcal{B}$ generates $\mathcal{C}(G)$.

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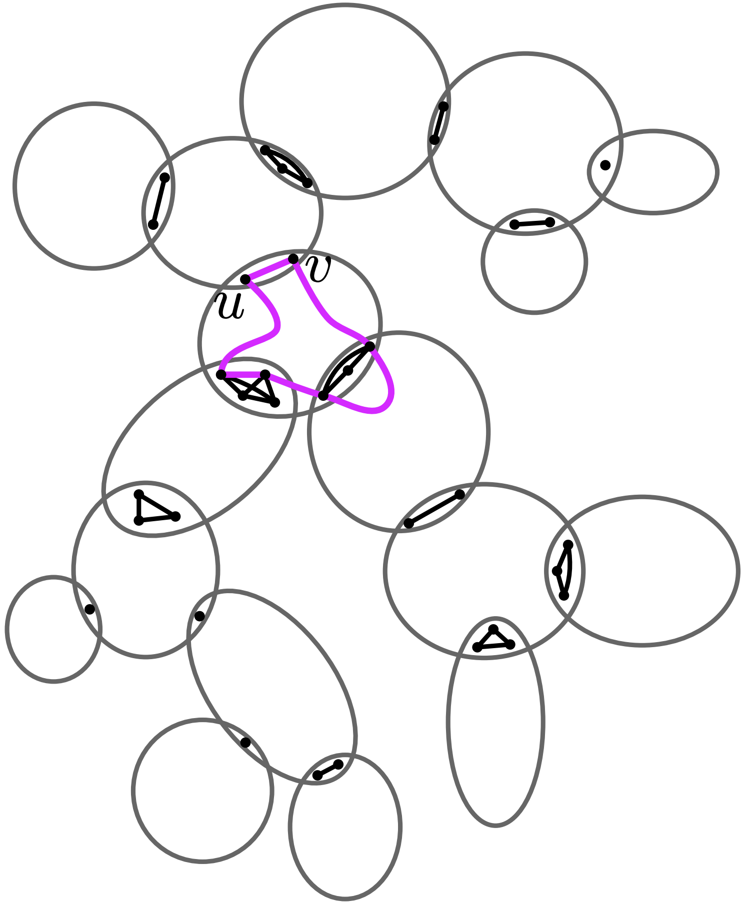

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What about the congestion?

Proof Attempt (cont.)

Torso $=$ a bag with the adhesions replaced by cliques.
($=$ a bag of G^+ in the previous slide.)

The previous ideas show:

Lemma

Let T be a tree decomposition of G whose torsos have basis number at most k.

For any each pair uv of vertices in each adhesion set, pick a uv-path P_{uv}. Suppose that the family $\{P_{uv}\}$ has congestion c.

Then $\mathrm{bn}(G) = O(ck)$.

An Extremely Complex Counter-Example

The important assumption was:

magic condition

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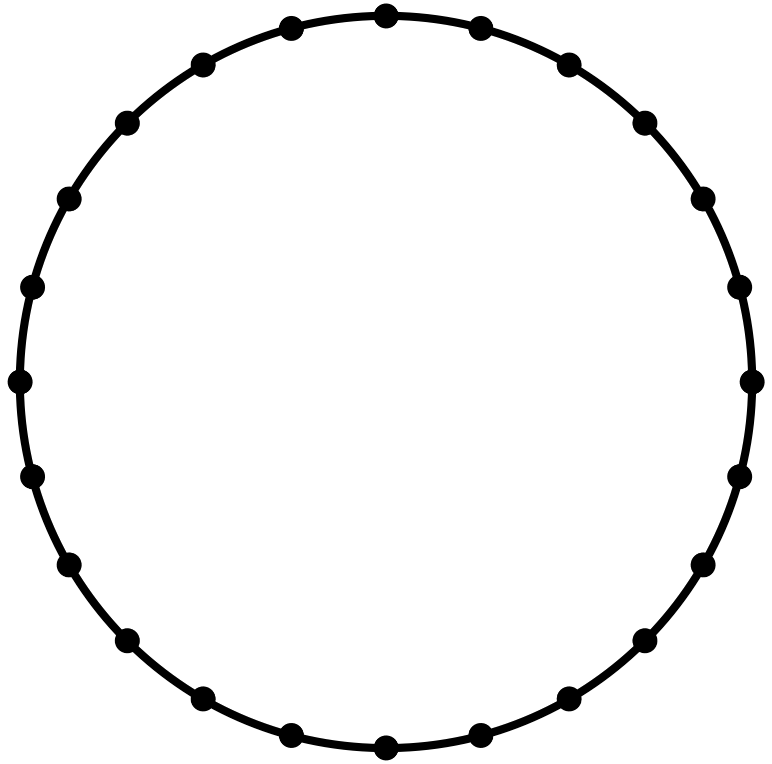

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many distance pairs must be in adhesions
$\Rightarrow$ large congestion.

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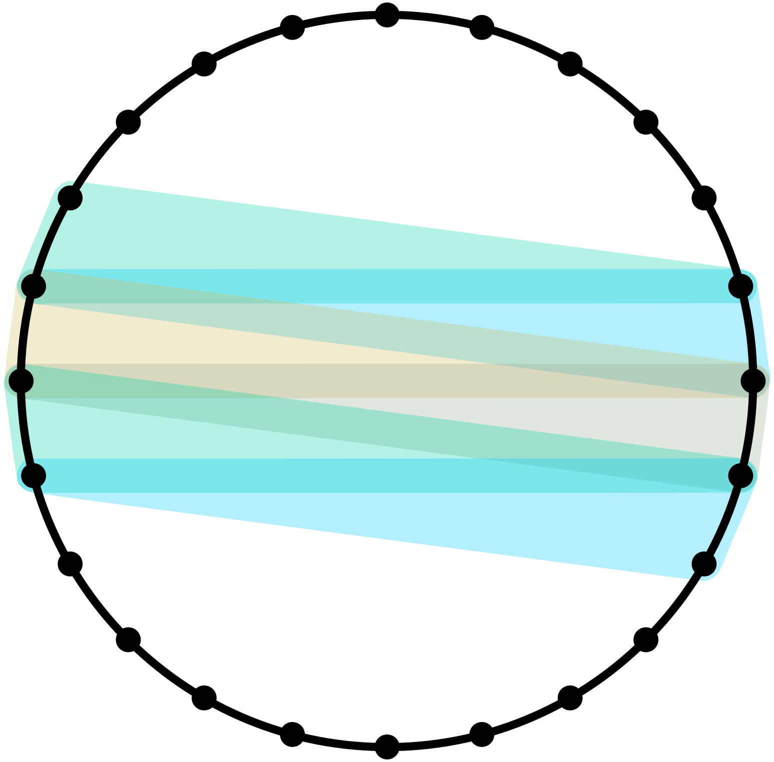

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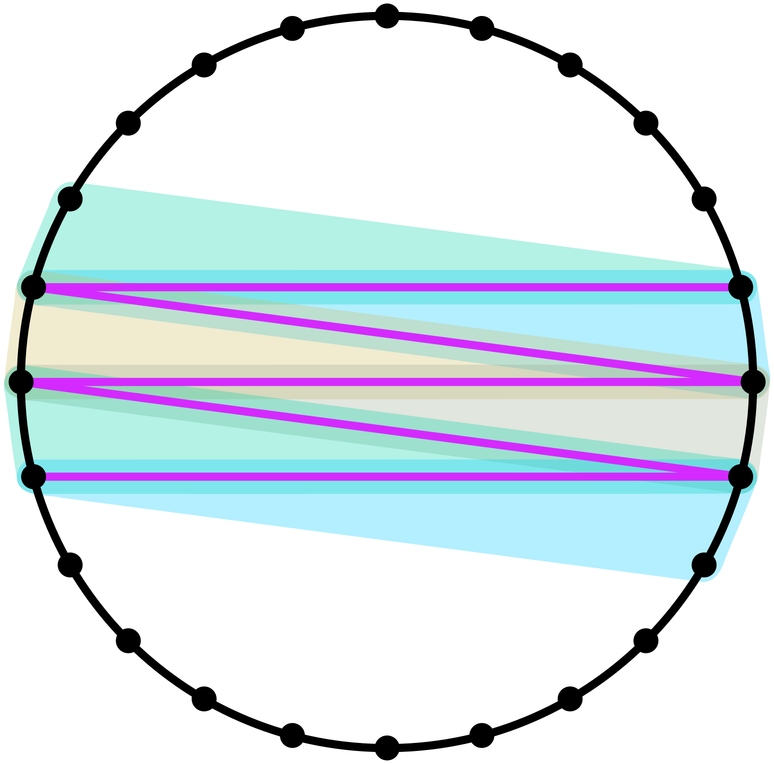

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Theorem (Bojańczyk & Pilipczuk '16)

Bounded tree-width graphs admit CMSO-transducible tree decompositions of bounded width.

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NOTHING TO DO WITH THIS TALK

Our key assumption is also helpful in their proof:

magic condition

For any each pair uv of vertices in each adhesion set, pick a uv-path P_{uv}. Suppose that the family $\{P_{uv}\}$ has congestion c.

Their proof works by reducing to cases where this assumption holds!

Bojańczyk & Pilipczuk — Proof Structure

Step 1

Assume G has tree-width k:

Lemma (Bojańczyk & Pilipczuk)

There is a tree-decomposition of G satisfying the magic assumption with congestion $O(k^4)$ and where torsos have path-width $O(k)$.

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Step 2

Assume G has *path-width* k:

Crazy decomposition based on the algebraic structure of finite semigroups
$\rightarrow \mathrm{bn}(G) = 2^{2^{O\left(k^2\right)}}$

Deus Ex Machina II: B. Miraftab, P. Morin, Y. Yuditsky

Theorem (Miraftab, Morin & Yuditsky '26)

Graphs of path-width k have basis number at most $4k$.

Proof: clever but elementary induction, adding vertices one-by-one.

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Using their result instead of the crazy decomposition of finite semigroups:

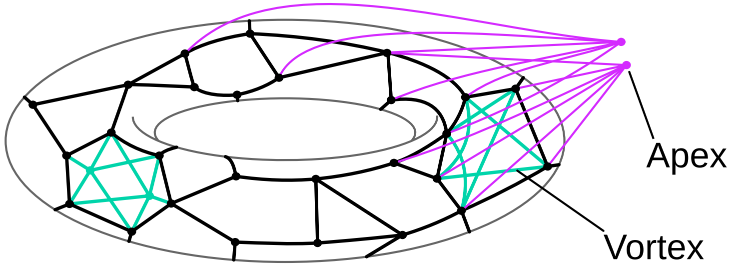
Theorem

Graphs of tree-width k have basis number $O(k^5)$.

Polynomial Bounds for Graph Minors

Theorem (Gorsky, Seweryn & Wiedderecht '25)

In Robertson & Seymour's Graph Minor Structure Theorem, for H-minor free graphs, all parameters can be chosen to be $O(|H|^{2300})$.



Polynomial Bounds for Graph Minors

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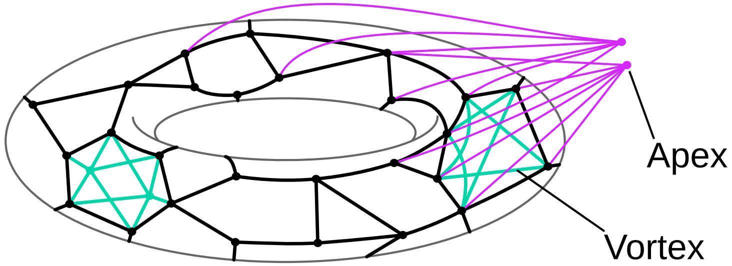

Corollary

H-minor free graphs have basis number $O(|H|^{32210})$.

Summary

Let $\mathcal{C}$ be a subgraph-closed class of graphs.

Theorem (G. & Giocanti)

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- take H with basis number $> k$
  $\Rightarrow$ $\mathcal{C}$ is H-minor free.

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Let $\mathcal{C}$ be a subgraph-closed class of graphs.

Theorem (G. & Giocanti)

$\mathcal{C}$ excludes a minor if and only if $\mathcal{C}$ has bounded basis number.

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  For pathwidth, treewidth, H-minor-free…
- Does bounded clique-width imply bounded basis number?
- Does excluding a vertex-minor imply bounded basis number???
- Say $\mathcal{C}$ is closed under induced subgraphs and has bounded basis number.
  Does this give some interesting structure?

Thank you!