Merge-width and χ-boundedness

Some elementary proofs with merge-width

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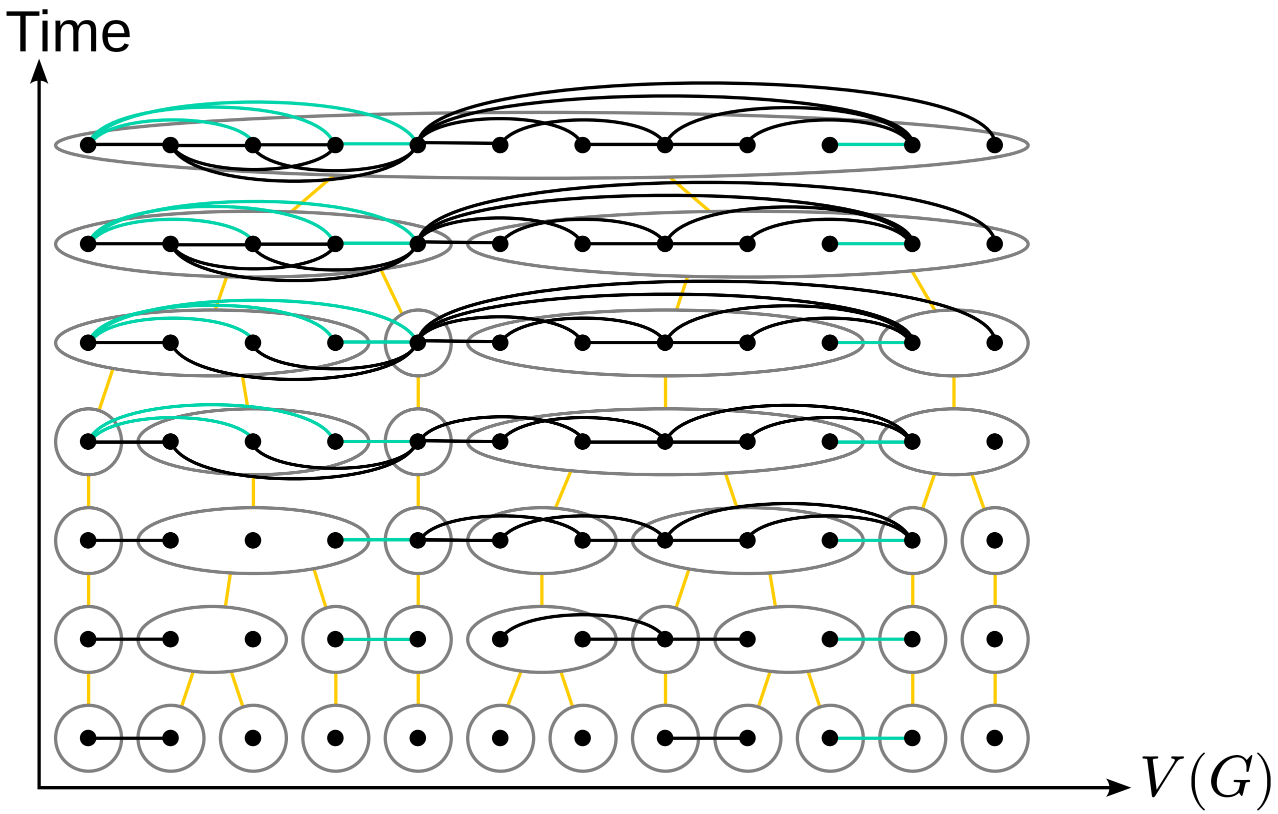
LoGAlg 2025

How to use a construction sequence?



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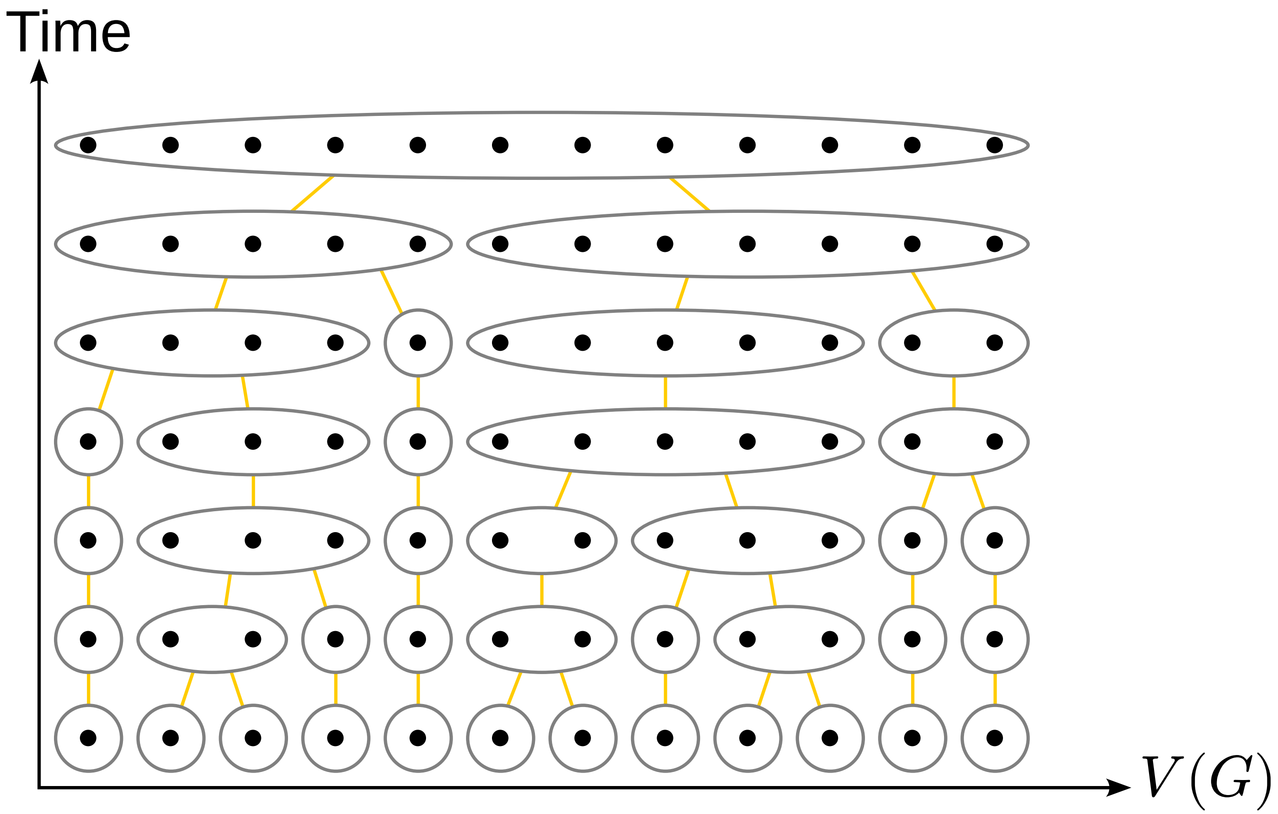

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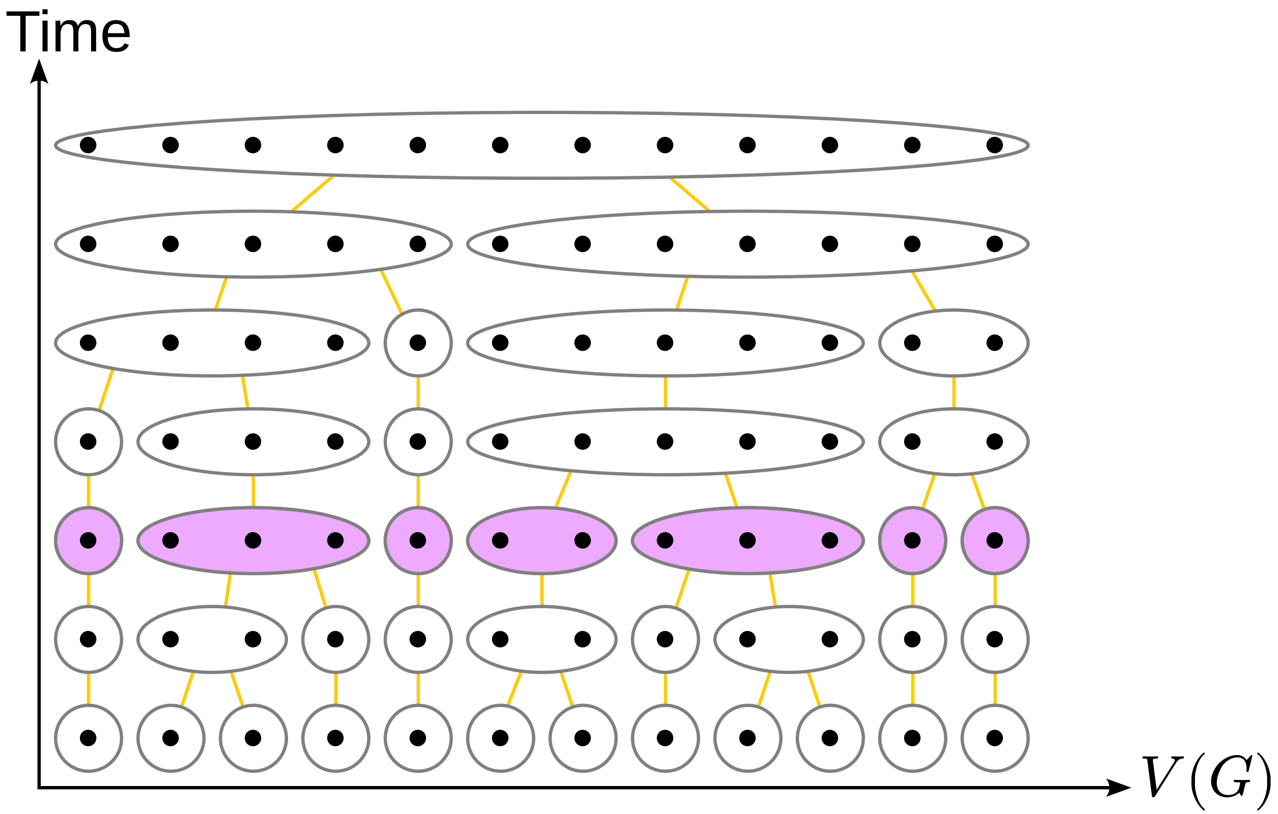

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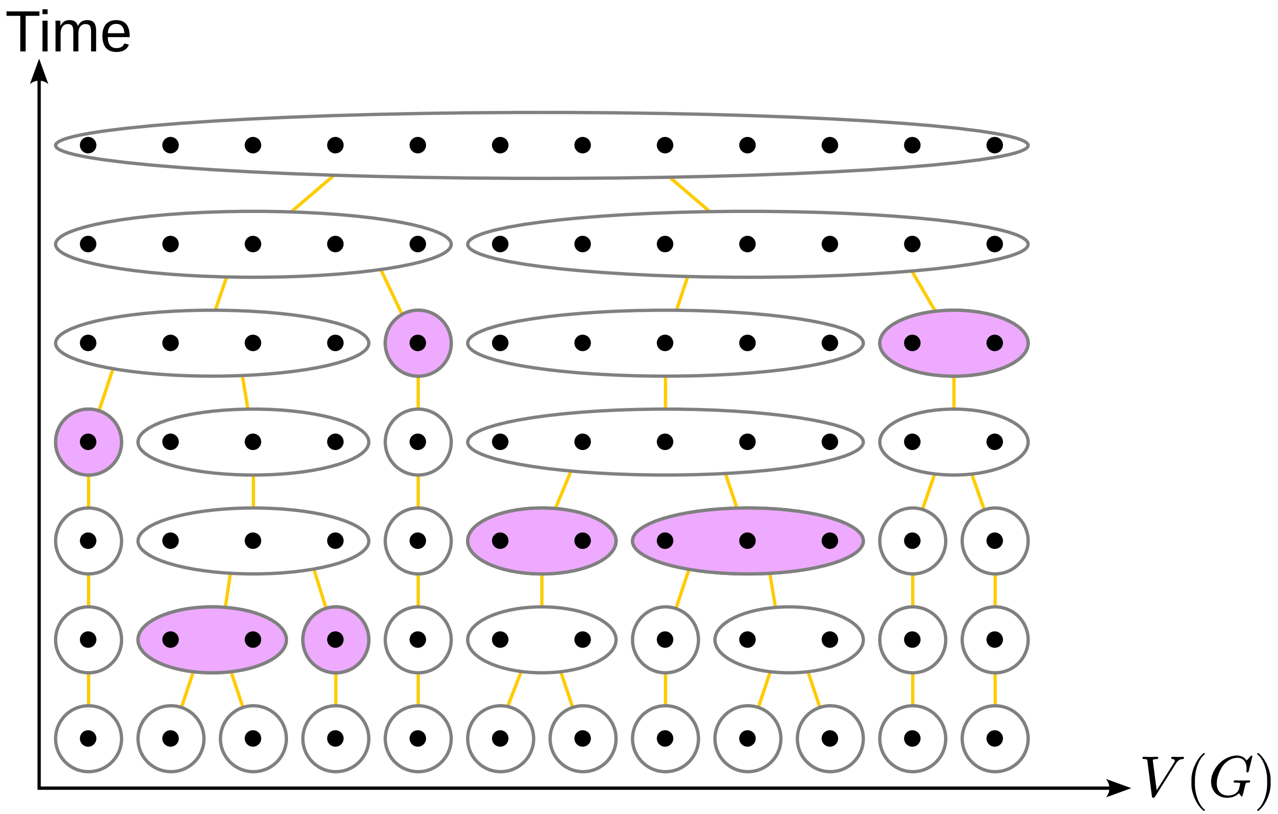

- For algorithms: dynamic programming over the whole sequence.
- For direct proofs: pick a single interesting step in the sequence,
- or pick a partition *across* $\mathcal{P}_1, ..., \mathcal{P}_m$ ("freezing").

Warmup: degeneracy

Theorem

Let G be a $K_{t,t}$-free graph with $\mathrm{mw}_1(G) \leq k$. Then G is $O(t^2 k)$-degenerate.

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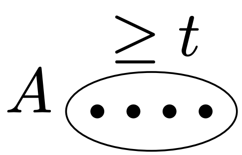
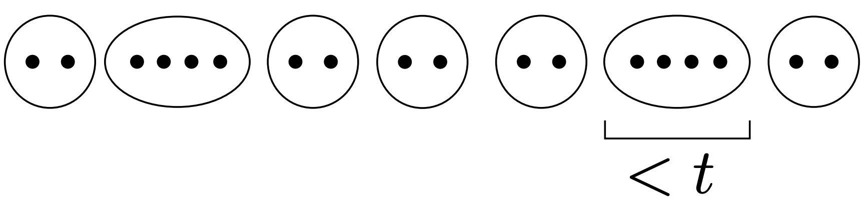
Take the first $\mathcal{P}_i$ with $A \in \mathcal{P}_i$ of size $\geq t$. So $|A| < 2t$, and other parts have size $< t$.





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(red edge = resolved edge/non-edge)

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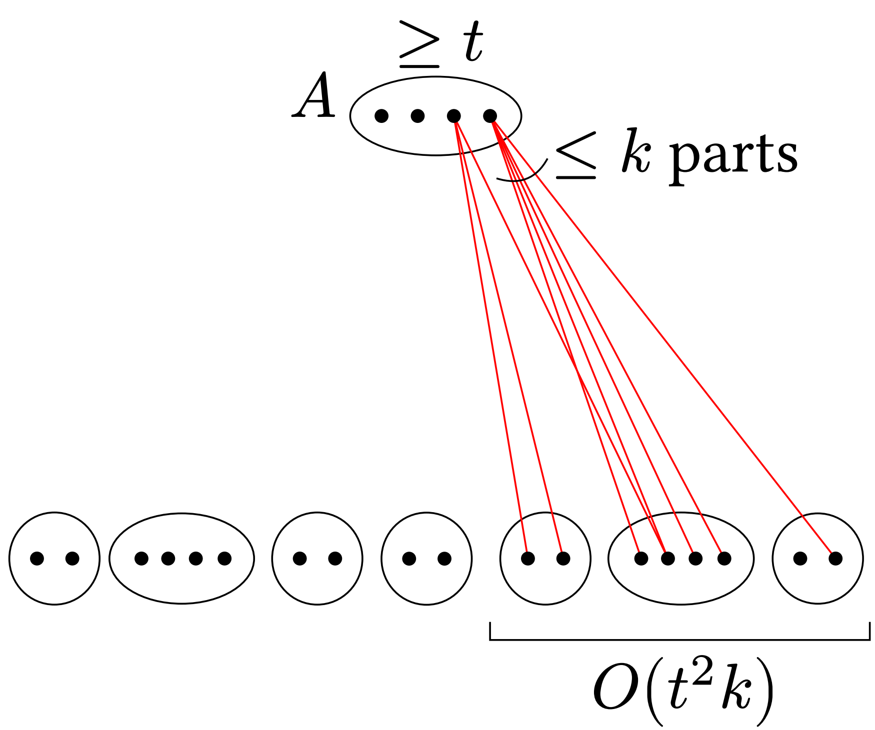

There are $\leq 2t^2k$ resolved pairs out of A. Call B the vertices not joined to A by any resolved pair.

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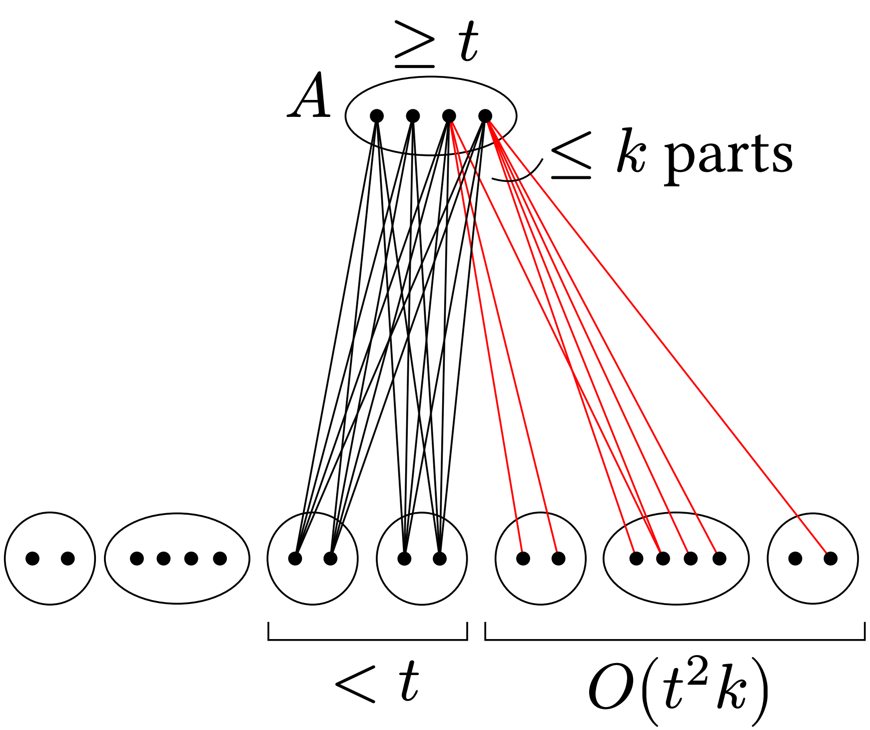


There are $\leq 2t^2k$ resolved pairs out of A. Call B the vertices not joined to A by any resolved pair.

Any $b \in B$ is either fully connected or non-adjacent to A. Less than t of them are fully connected to A.

χ-boundedness

$\chi(G)$: minimum number of colours to properly colour G

$\omega(G)$: maximum clique size in G

Goal: bounded merge-width implies χ-bounded.

Theorem

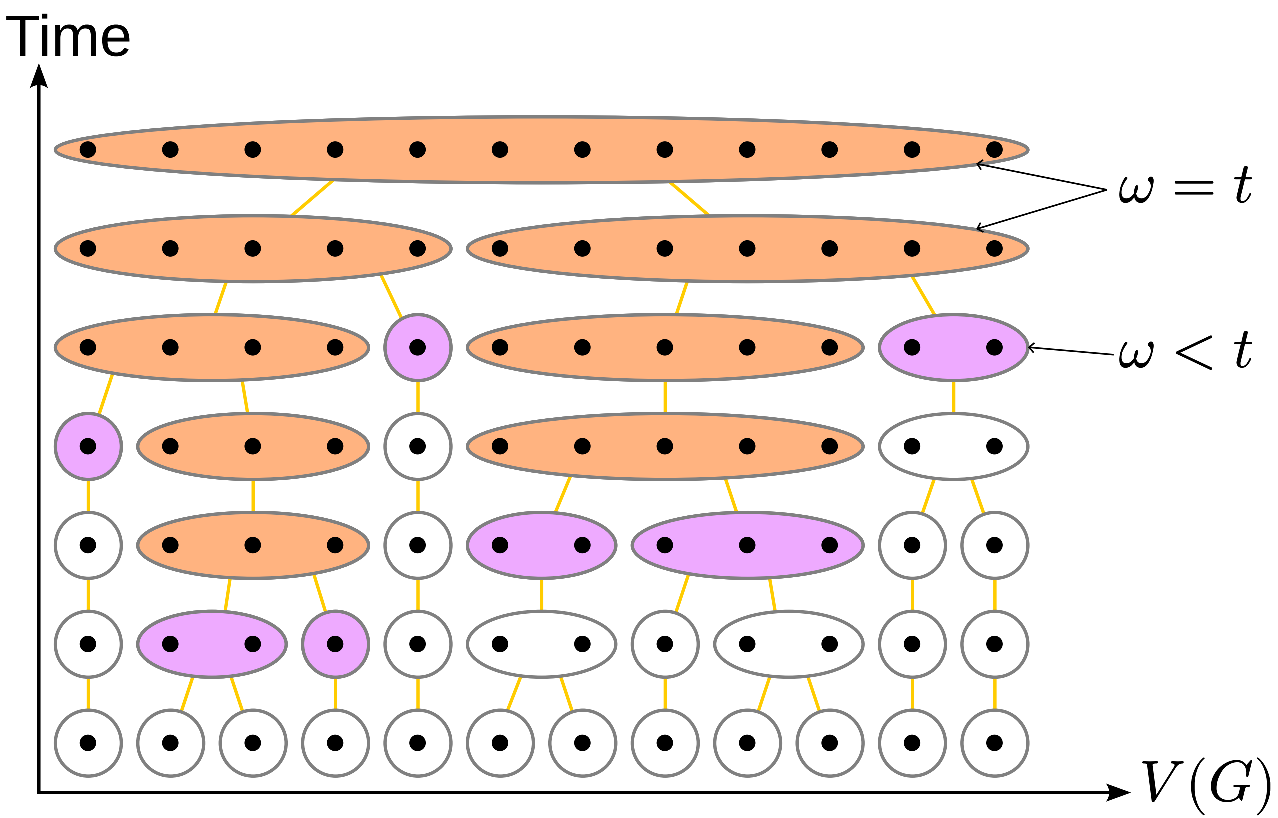
There is a function f such that any graph G satisfies

$$\chi(G) \leq f(\omega(G), \mathrm{mw}_2(G)).$$

We fix $k := \mathrm{mw}_2(G)$, $t := \omega(G)$.

Decreasing ω

$(t := \omega(G))$



Across all partitions $\mathcal{P}_1, \ldots, \mathcal{P}_m$, take the maximal parts P such that $\omega(G[P]) < t$.

This is a partition $\mathcal{P}$.

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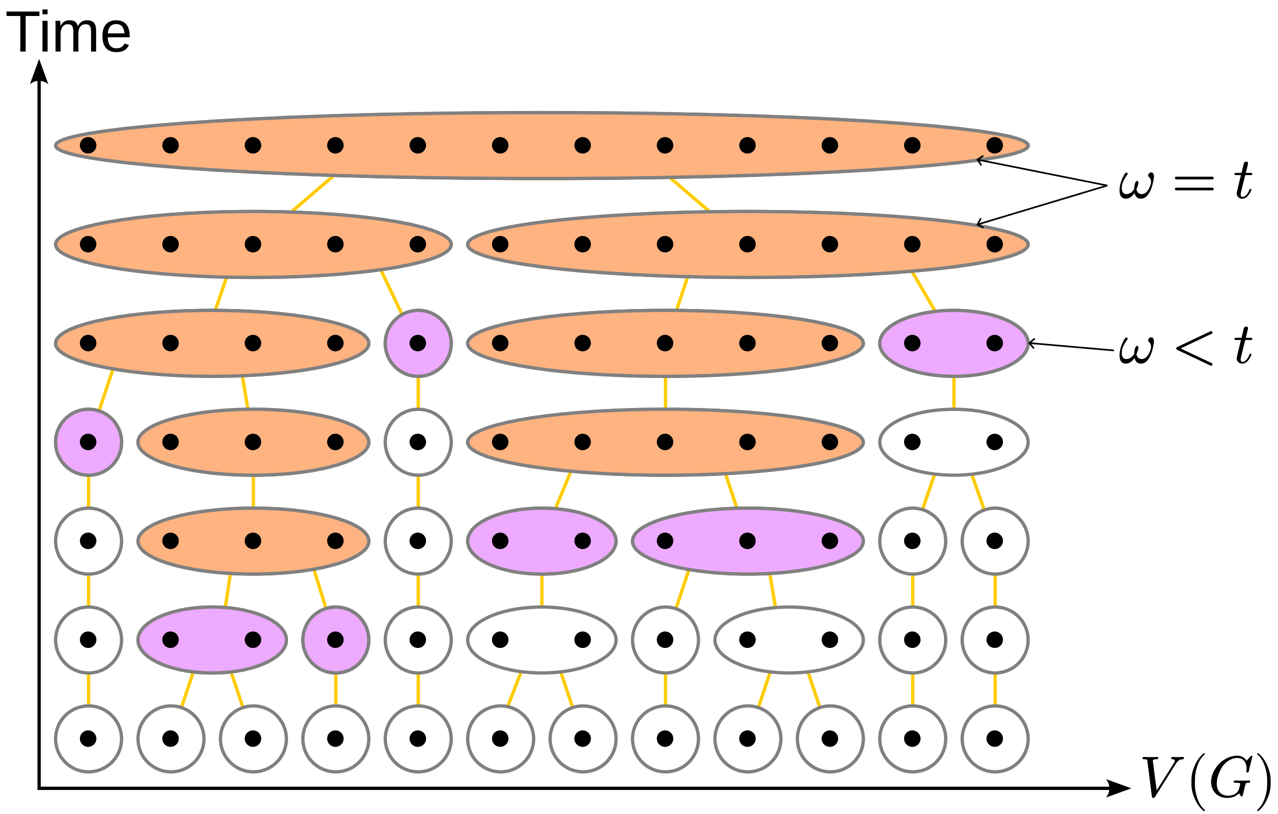

Across all partitions $\mathcal{P}_1, ..., \mathcal{P}_m$, take the maximal parts P such that $\omega(G[P]) < t$.

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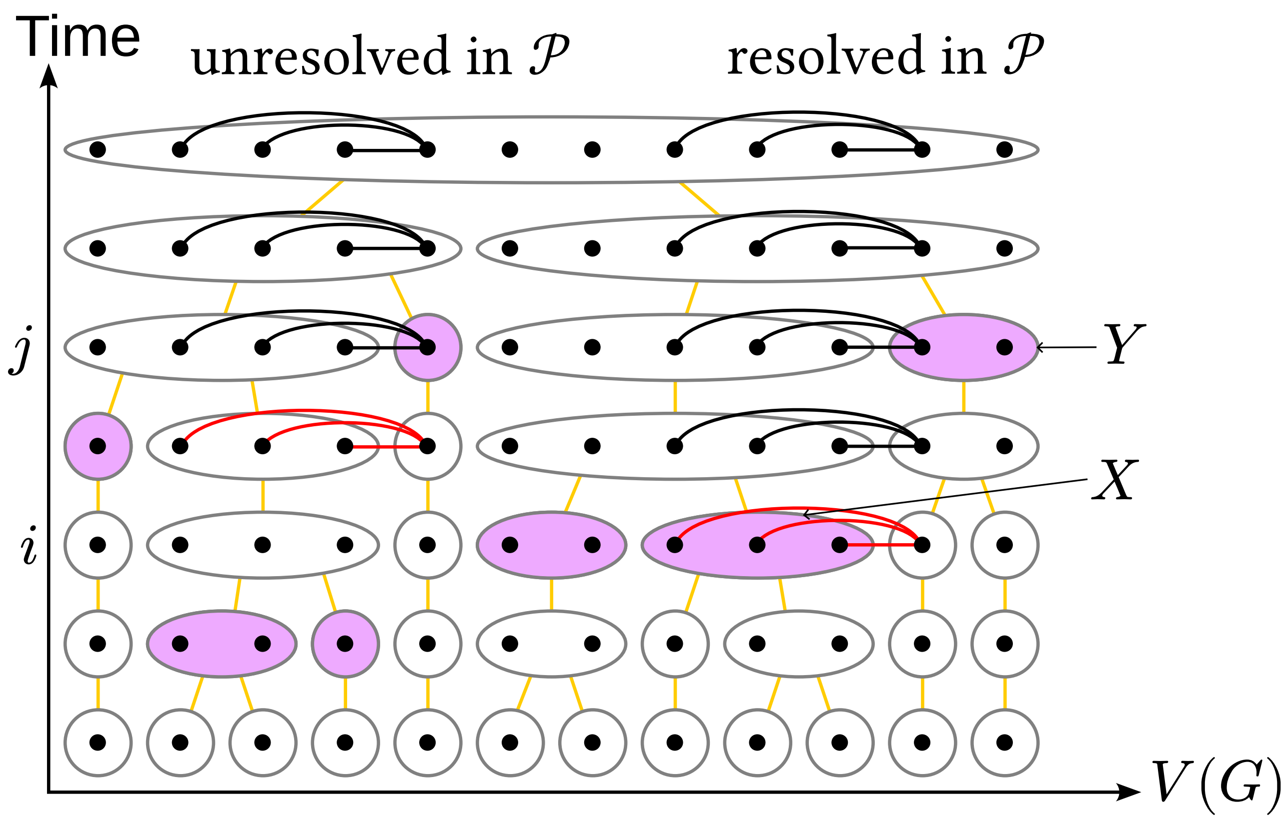
By induction on ω, we can colour each $P \in \mathcal{P}$, so we focus on edges between distinct parts.

New and Old edges



Take an edge xy between parts $X \neq Y$ in $\mathcal{P}$. Say X, Y come from $\mathcal{P}_i, \mathcal{P}_j$.

We say xy is *resolved in* $\mathcal{P}$ if xy was resolved at time $\min(i, j)$, or earlier.

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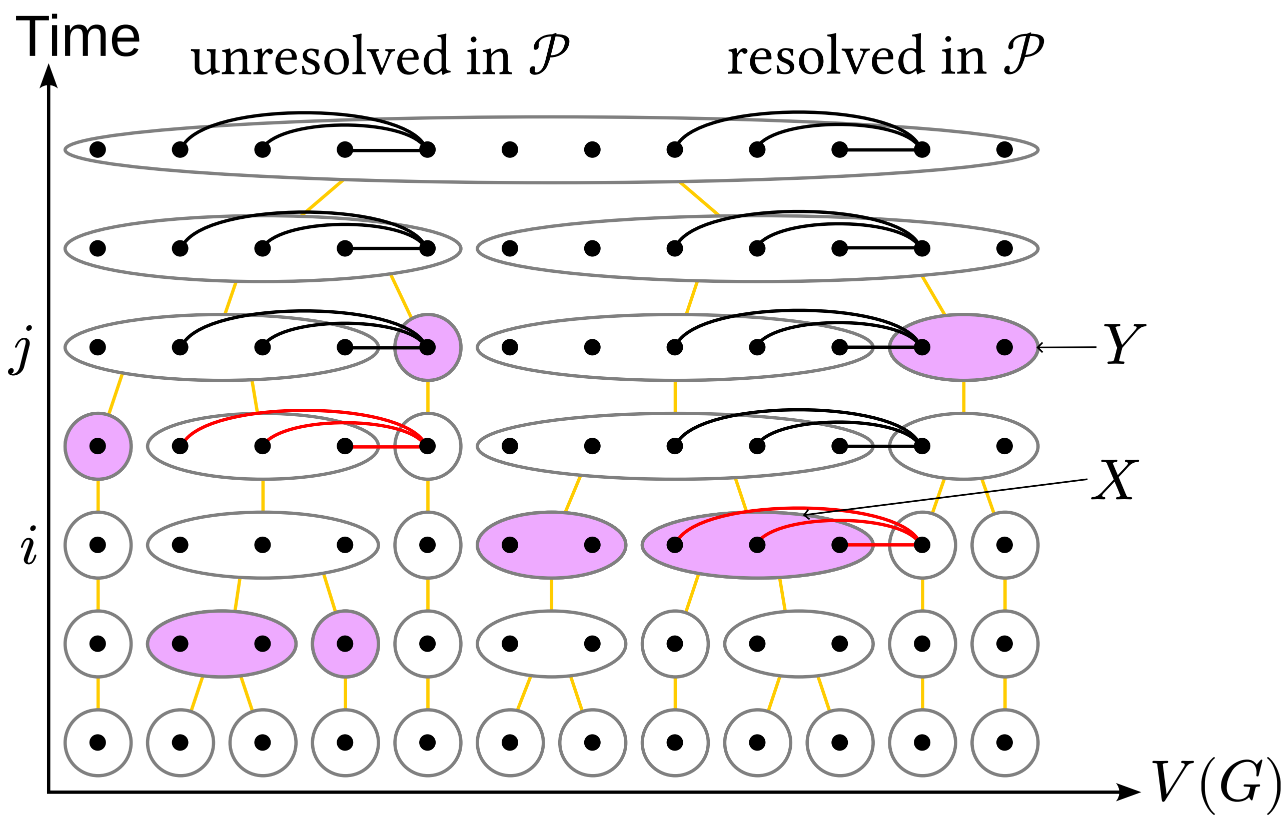

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Let E_R, E_U be the resolved/unresolved edges in $\mathcal{P}$.
We colour (V, E_R) and (V, E_U) independently.

Product colouring

Summary:

We have three kinds of edges associated with $\mathcal{P}$:

- edges inside a part $P \in \mathcal{P}$
- E_R: edges created by resolving between parts in or before $\mathcal{P}$ ("resolved in $\mathcal{P}$")
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We colour each of the 3 edge set separately, and combine with a product colouring.

For edges inside a part of $\mathcal{P}$: use induction on ω.
Now we deal with E_U, and then E_R.

Unresolved edges in $\mathcal{P}$

$k := \mathrm{mw}_2(G)$, $t := \omega(G)$

Order parts of $\mathcal{P}$ by the time they are merged with something else.

Claim

Each $P \in \mathcal{P}$ has *unresolved* edges to most kt *later* parts in $\mathcal{P}$.

In other words, $(V, E_U)/\mathcal{P}$ is kt-degenerate, and (V, E_U) is $(kt + 1)$-colourable.

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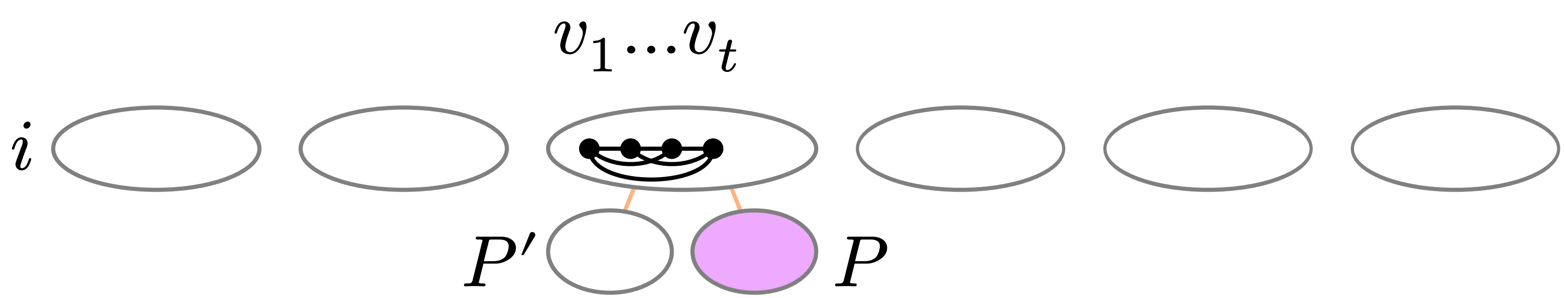

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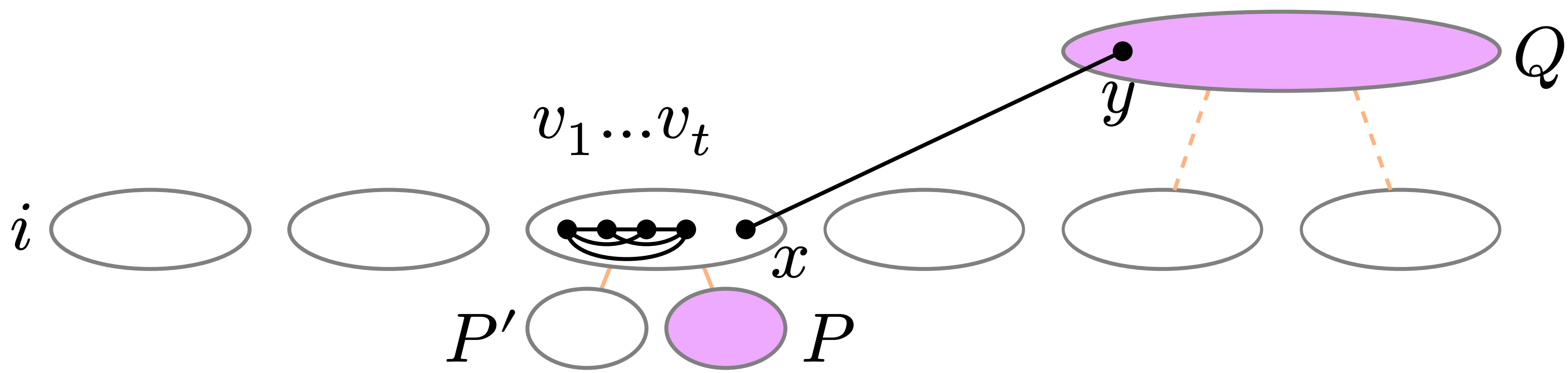

Say P is merged with P' at time i. Then $G(P \cup P')$ contains a clique $\{v_1, ..., v_t\}$.
Say $Q \in \mathcal{P}$ disappears after time i, and $xy \in E_U$ is an edge between P and Q.

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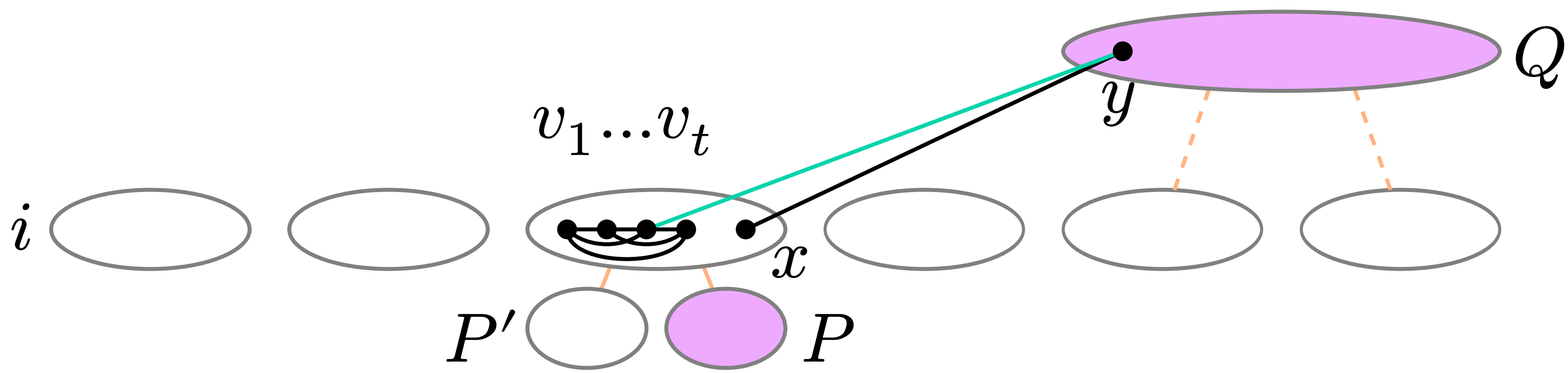

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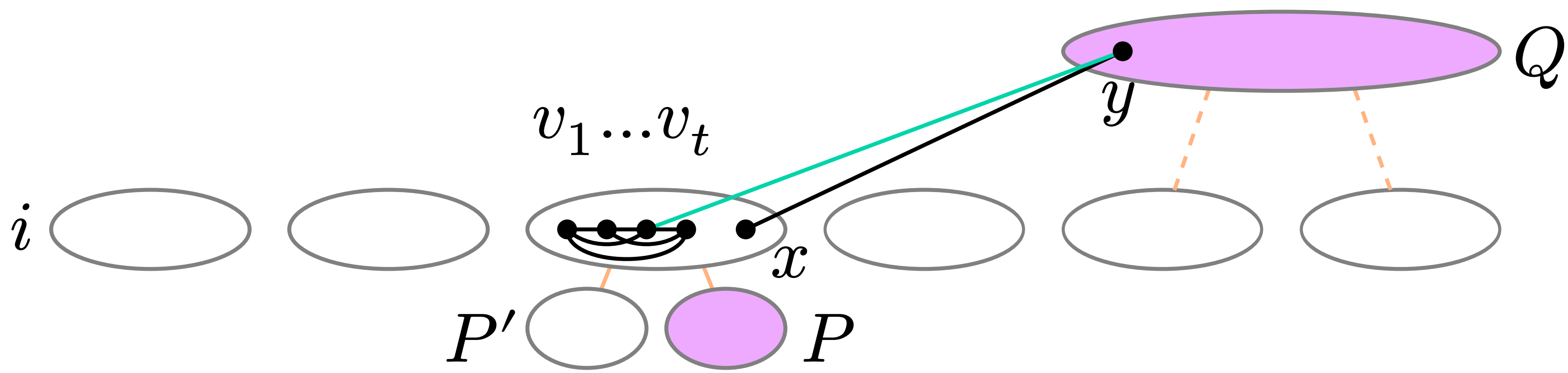

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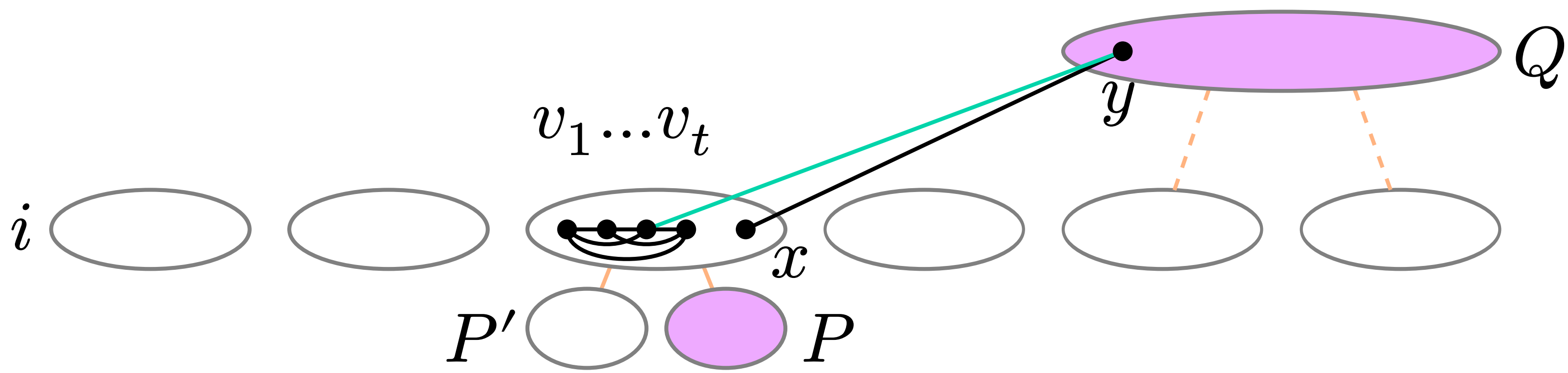

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Since xy is in E_U, it is unresolved at time i. So $v_i y$ must be resolved at time i.

This leaves only kt choices for Q (k choices for each v_i).

Resolved edges in $\mathcal{P}$

Reminder:

- We removed edges inside each $P \in \mathcal{P}$, so $\mathcal{P}$ is a partition into independent sets.
- Each edge $xy \in E_R$ is created by resolving between parts X, Y in or before $\mathcal{P}$. In particular, X, Y are independent sets in E_R.

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Thus for $G_R := (V, E_R)$, we have a merge sequence satisfying the following:

($*$) Parts X, Y can only be positively resolved if X, Y are independent sets in G_R.

Lemma

If G_R has a construction sequence satisfying ($*$), and with radius-2 width k, then $\chi(G_R) \leq k$.

Resolving only independent sets

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Lemma

If G_R has a construction sequence satisfying ($*$), and with radius-2 width k, then $\chi(G_R) \leq k$.

Sketch:

We make a partition from the merge sequence once again!
$\mathcal{P} :=$ maximal parts that are positively resolved with something.

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Sketch:

We make a partition from the merge sequence once again!
$\mathcal{P} :=$ maximal parts that are positively resolved with something.

For any $P \in \mathcal{P}$:

(1) P is an independent set by (∗).

(2) P has radius 1 in the graph of resolved edges,

(3) and (2) implies that $G_R/\mathcal{P}$ is k-degenerate.

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We colour each of the 3 edge set separately, and combine with a product colouring:

- edges inside parts of $\mathcal{P}$: induction on ω
- E_U: degeneracy argument, uses that parts $P \in \mathcal{P}$ are maximal without K_t
- E_R: have a construction sequence where we only resolve between independent sets. Uses a second freezing + degeneracy argument.

Open questions

- Is bounded $\mathrm{mw}_1(G)$ enough for χ-boundedness?
- Polynomial χ-boundedness?
- Extend to bounded flip-width, linear neighbourhood complexity,…

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Thank you!