Factorising Pattern-Free Permutations

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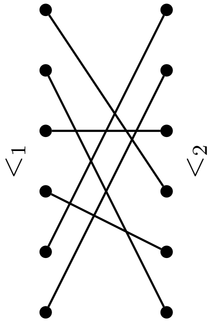
ENS Lyon

SODA 24

Permutations

Permutation = two linear orders on the same set: $(X, <_1, <_2)$



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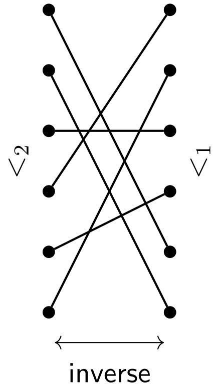

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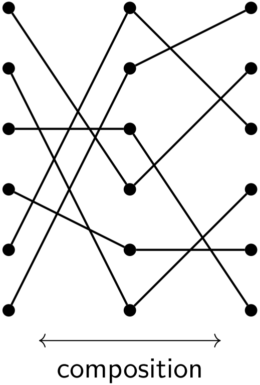
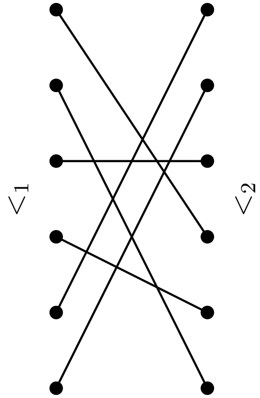


Patterns in permutations



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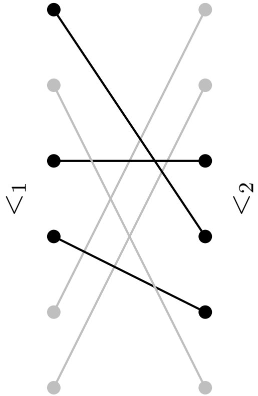

Permutation $(X, <_1, <_2)$

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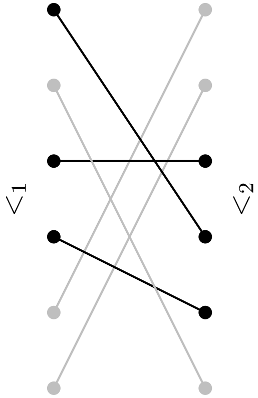

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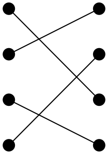
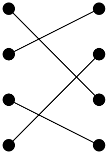
Pattern-free permutation class:

$$\mathcal{F}(\tau) = \{\sigma \ : \ \tau \not\subseteq \sigma\}$$

Separable permutations

Separable permutations
$= \mathcal{F}(3142, 2413)$





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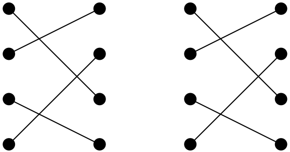

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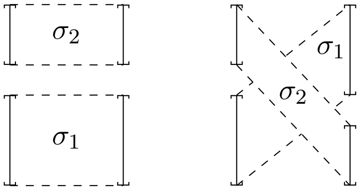

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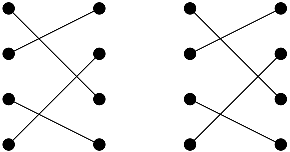

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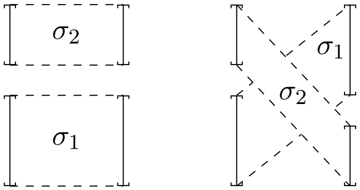

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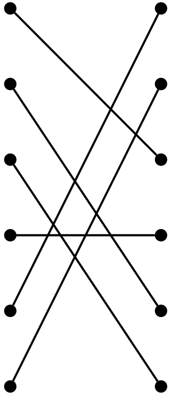

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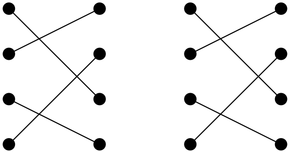

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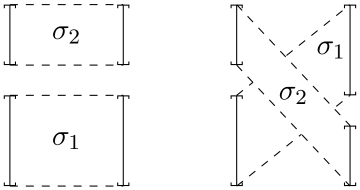

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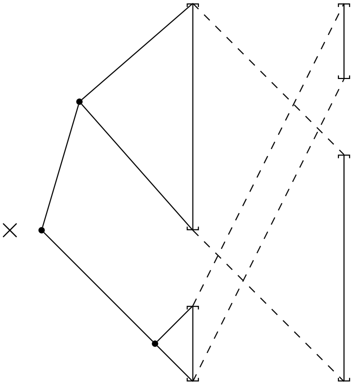

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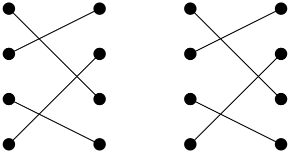

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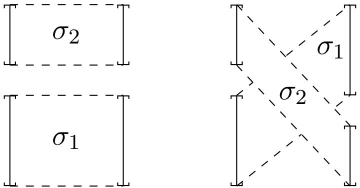

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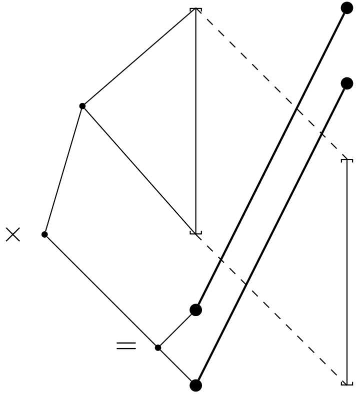

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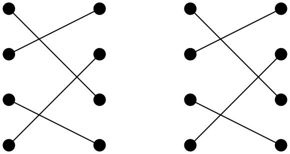

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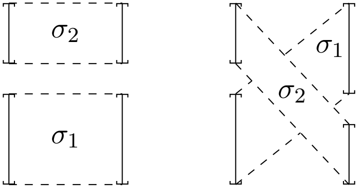

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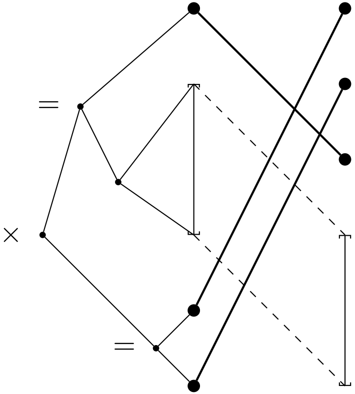

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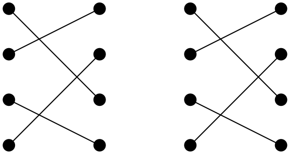

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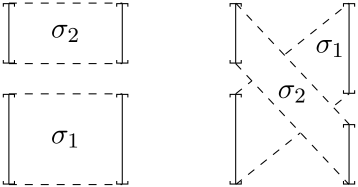

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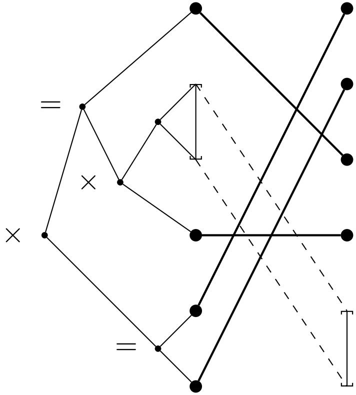

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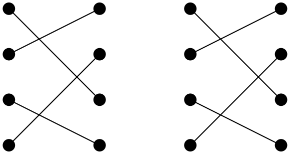

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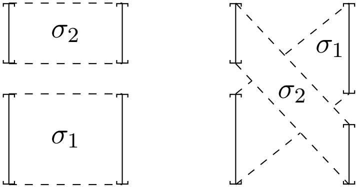

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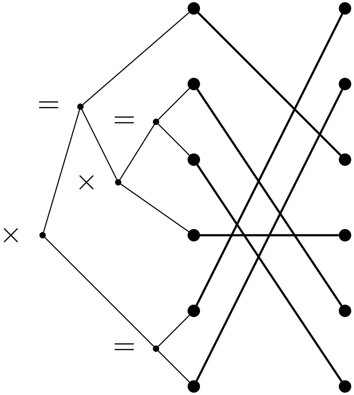

Pattern-free classes are nice

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For any τ, there is a constant c such that $\mathcal{F}(\tau)$ has $\leq c^n$ permutations of size n.

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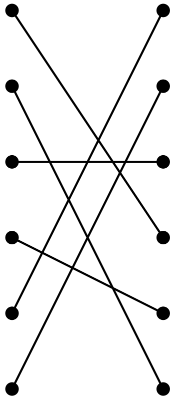
Twin-width

Twin-width of $(X, <_1, <_2)$:

- iteratively merge elements of X
- error between $A, B \subset X$ if they interleave for either $<_1$ or $<_2$
- minimize the error degree



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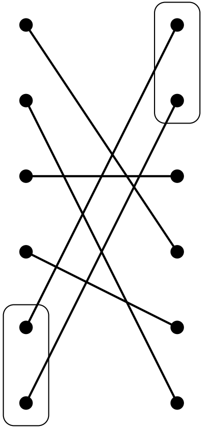

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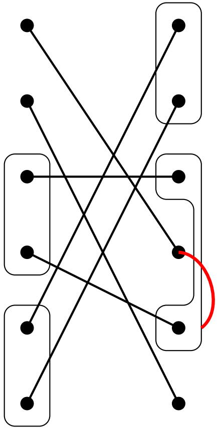

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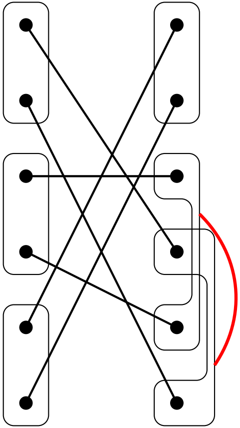

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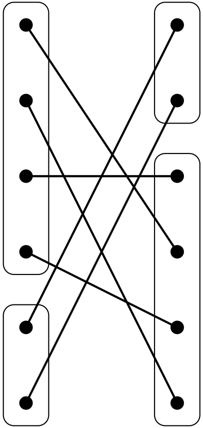

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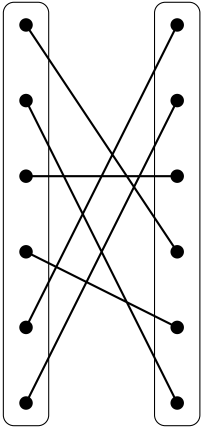

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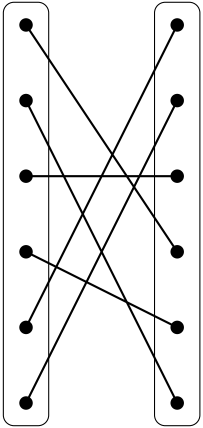
Separable $\iff$ twin-width $= 0$



Guillemot–Marx Algorithm

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Win–win argument:

Lemma

A class $\mathcal{C}$ avoids a pattern if and only if it has bounded twin-width.

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One can test if τ is a pattern of σ in time $f(\tau, \mathrm{tww}(\sigma)) \cdot |\sigma|$.

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We give a 'decomposition':

Theorem (BBGT)

For any pattern τ, there is a constant k such that any $\sigma \in \mathcal{F}(\tau)$ factorises as $\sigma = \sigma_1 \circ \cdots \circ \sigma_k$, with σ_i separable.

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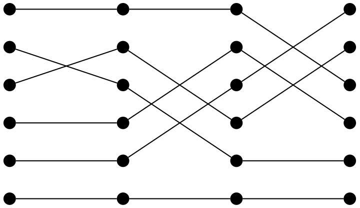

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For any σ_1, σ_2, $\operatorname{tww}(\sigma_1 \circ \sigma_2) \leq f(\operatorname{tww}(\sigma_1), \operatorname{tww}(\sigma_2))$.

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Corollary

For a class $\mathcal{C}$ of permutations, TFAE:

- *$\mathcal{C}$ avoids a pattern,*
- *$\mathcal{C}$ has bounded twin-width,*
- *$\mathcal{C} \subset \mathcal{S}^k$ for some $k \in \mathbb{N}$ ($\mathcal{S}$ = separable permutations).*

A class $\mathcal{C}$ of permutations avoids a pattern if and only if $\mathcal{C} \subset \mathcal{S}^k$ for some $k \in \mathbb{N}$.

Proof overview (from very far away)

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For permutations, this decomposition can be expressed with direct and skew sums, and a bounded number of products.

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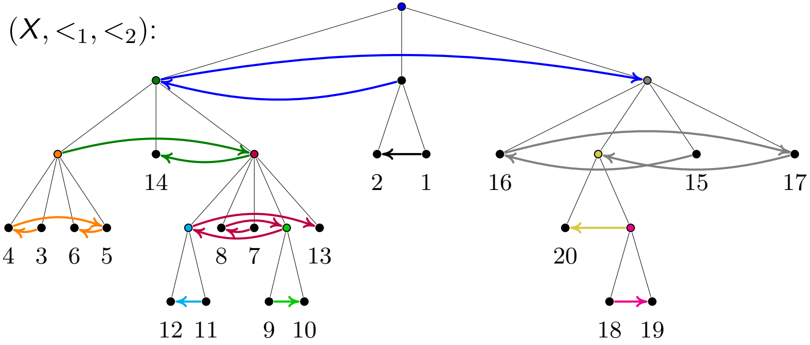

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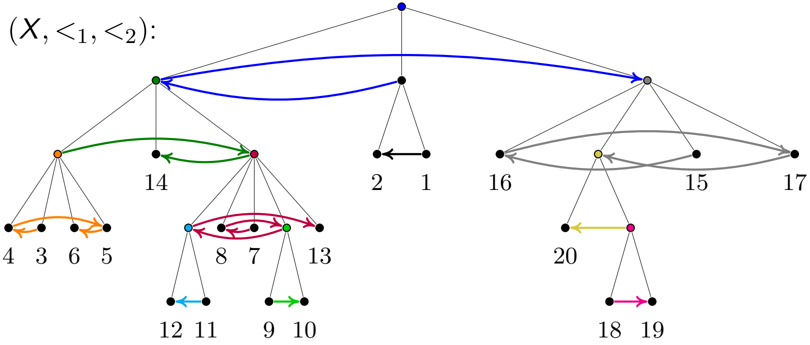

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Lemma

If all local permutations are separable, so is the global permutation.

Delayed Substitution

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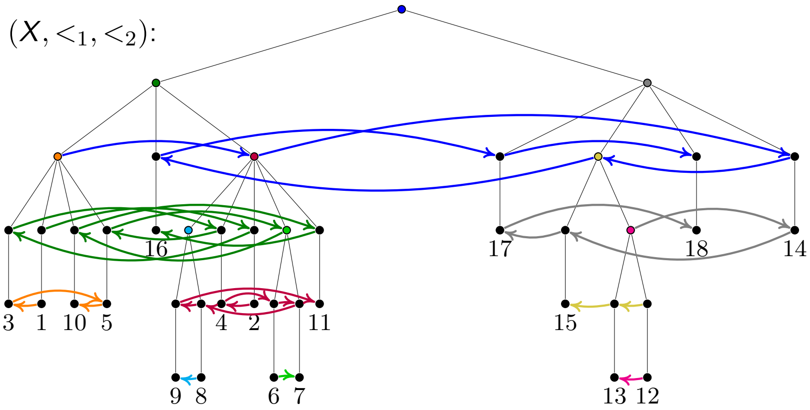

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Using Delayed Substitutions

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To factorise a permutation σ:

- compute the delayed substitution for σ,
- recursively factorise the local permutations,
- rewrite into composition of substitution of separable permutations (using some distributive property)

Application to Graphs

Corollary (Sparse graphs)

There are $f : \mathbb{N} \to \mathbb{N}$ and $c \in \mathbb{N}$ satisfying the following: if G has no $K_{t,t}$-subgraph and $\mathrm{tww}(G) \leq k$, then the $f(k,t)$-subdivision of G has twin-width $\leq c$.

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Conjecture: $c = 4$ can be reached for both results.

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